AD-A275 360

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

2. REPORT DATE Dec 93	3. REPORT TYPE AND DATES COVERED THESIS/DISSERTATION
4. TITLE AND SUBTITLE Stability Properties of Inclusive Connectivity for Graphs	5. FUNDING NUMBERS
6. AUTHOR(S) DAVID W. Cribb	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) AFIT Student Attending: Clemson Univ	8. PERFORMING ORGANIZATION REPORT NUMBER AFIT/CI/CIA- 93-29D
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) DEPARTMENT OF THE AIR FORCE AFIT/CI 2950 P STREET WRIGHT-PATTERSON AFB OH 45433-7765	10. SPONSORING/MONITORING AGENCY REPORT NUMBER
11. SUPPLEMENTARY NOTES	
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for Public Release IAW 190-1 Distribution Unlimited MICHAEL M. BRICKER, SMSgt, USAF Chief Administration	12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

DTIC
ELECTE
FEB 04 1994
S A

94-03990

94 2 03 204

14. SUBJECT TERMS	15. NUMBER OF PAGES 128
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT

STABILITY PROPERTIES OF INCLUSIVE CONNECTIVITY FOR GRAPHS

A Dissertation

Presented to

the Graduate School of

Clemson University

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

Mathematical Sciences

---

by

David W. Cribb

December 1993

December 9, 1993

To the Graduate School:

This dissertation entitled "Stability Properties of Inclusive Connectivity for Graphs" and written by David W. Cribb is presented to the Graduate School of Clemson University. I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy with a major in Mathematical Sciences.

Richard D. Ringeisen

Dissertation Advisor

We have reviewed this dissertation and recommend its acceptance:

Renu Laskar

[illegible]

William R. Hare

Warren P. Adams

Accesion For		
NTIS CRA&I	☑	
DTIC TAB	☐	
Unannounced	☐	
Justification		
By		
Distribution /		
Availability Codes		
Dist	Avail and / or Special	
A-1		

DTIC QUALITY INSPECTED 8

Accepted for the Graduate School:

Calvin M. Lawson

STABILITY PROPERTIES OF INCLUSIVE CONNECTIVITY FOR GRAPHS

A Dissertation

Presented to

the Graduate School of

Clemson University

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

Mathematical Sciences

---

by

David W. Cribb

December 1993

ABSTRACT

This dissertation is an investigation of inclusive connectivity which is a localization of connectivity defined for each vertex and each edge of a graph. The inclusive edge (vertex, mixed) connectivity of a vertex v is the minimum number of edges (vertices, graph elements) whose removal yields a subgraph in which v is a cutvertex. All possible combinations of these three parameters with regard to edge addition stability, in which the value of the parameter will remain unchanged after the addition of any edge, is studied along with other various properties including a relationship between the stability of inclusive connectivity and global connectivity. A similar study in the stability for inclusive connectivity for edge deletion is conducted. Final topics include neutral edges, where a neutral edge is one whose removal does not change the respective inclusive connectivity value of any vertex, and inclusive connectivity stable graphs, where the sum of the respective inclusive connectivity values for all vertices remains the same no matter what edge is deleted.

ACKNOWLEDGEMENTS

I would like to thank Dr. Richard Ringeisen for his guidance, ideas and encouragement through my entire graduate study. I would also like to thank all my committee members, especially Dr. James Boland. I gratefully acknowledge the sponsorship of graduate study by the Mathematical Sciences Department of the United States Air Force Academy and the financial support of the United States Air Force and the Office of Naval Research. Finally, I would like to dedicate this dissertation to my family, Alexia and Adam, whose love and support made this all possible.

TABLE OF CONTENTS

Table of Contents (Continued)

LIST OF TABLES

LIST OF FIGURES

List of Figures (Continued)

List of Figures (Continued)

List of Figures (Continued)

CHAPTER 1

PREVIEW

This dissertation deals with the three inclusive connectivity parameters and how these parameters change in relation to various graph operations and with respect to different graph elements. Inclusive connectivity is a type of "local connectivity" parameter defined for each vertex and edge of a graph.

The introduction of inclusive connectivity by Lipman and Ringeisen in 1979 resulted from an application to alliance graphs. Given a set of countries with specified alliances between pairs of countries, inclusive connectivity answered the question "How close can a particular country come to severing the alliance connections between two groups of countries?" Intuitively, the "stress" on a vertex is raised or lowered according to its inclusive connectivity being lowered or raised, respectively [22].

Similar applications include communication networks, supply and delivery systems, and transportation networks where one may like to know how much stress or vulnerability is placed on a center (node) after the destruction or creation of a specified link (edge).

Inclusive connectivity can be conceptualized as the required inclusion of a given vertex or edge in a minimum separating set which requires that vertex or edge for disconnection. While this dissertation deals with many varied topics concerning inclusive connectivity, it provides the first in-depth investigation of the effects on inclusive connectivity during edge addition (creation). The topics are organized into chapters of a homogeneous nature which we will now briefly describe.

Chapter 2 includes all necessary definitions and explanations of notation used in this document. Several examples are presented to acquaint the reader with the

fundamental ideas of inclusive connectivity as well as several well known graph theory results. A complete literature review is presented detailing the history and development of inclusive connectivity. Extensions of several of these fundamental results are established for all the parameters. The last topic is an analysis of inclusive connectivity for edges, as suggested by Boland in [2].

A detailed study of the changes in inclusive connectivity after edge addition is contained in Chapter 3. The largest section of this dissertation begins with many examples of the possible relationships among the inclusive connectivity parameters for vertex stability under edge addition. Each of these examples demonstrates that an infinite class of such graphs exist. Numerous results are presented describing graphs when inclusive connectivity stability is known. This section also uses inclusive connectivity to answer an open problem proposed in [2] regarding a relationship between two of the parameters. Finally, an important result shows a relationship between the stability of inclusive connectivity and the stability of the global connectivities, under edge addition.

Chapter 4 presents an alternative from [18] on the stability of inclusive connectivity under edge deletion. This alternative allows, for two parameters, a definition of inclusive connectivity stability under edge deletion for a single vertex as opposed to a global definition. Many extensions of previous work by Ringeisen, Lipman, and Rice for a single parameter are presented, leading to a characterization concerning stability in the remaining two parameters.

Chapter 5 has three main topics. The first two sections deal with an inclusive connectivity neutral edge. A neutral edge is an edge whose removal does not alter the inclusive connectivity value for any vertex in the graph. Infinite classes of graphs illustrate combinations of every possible type of neutrality for an edge. Next, we present a surprising result on the change in the total number of neutral edges in a graph after the deletion of a neutral edge. Finally, several examples of different types

of stable graphs are illustrated. These graphs are referred to as stable because the sum of the inclusive connectivity values for all vertices remains unchanged after the deletion of an arbitrary edge.

Chapter 6 presents several conjectures on open problems and ideas for future research involving inclusive connectivity. Included are several extensions of an extremely useful theorem from [2] which could prove helpful in the resolution of these conjectures.

CHAPTER 2

PRELIMINARIES

Introduction

This chapter contains a review of the notation and definitions used in this study of inclusive connectivity parameters. These parameters are defined for both vertices and edges, and can be regarded as measures of how close a vertex or edge is to being a cutvertex or bridge respectively. Several examples are included to help the reader visualize the concepts. Any definitions or notation not specified here can be found in [7].

Also included are several well known graph theory results, which are fundamental for several parts of this study, and are included without proof. (For such, see [7].)

Inclusive connectivity was introduced first as cohesion by Lipman and Ringeisen [14]. We review all subsequent research resulting from this introduction and several extensions of the results obtained in [22] for the cohesion parameter, $\lambda_i(v)$. Previous results for cohesion, for the most part, can be extended to include all the inclusive connectivity parameters.

We conclude this chapter by examining inclusive connectivity for edges and how that can be related to inclusive connectivity for vertices.

Definitions and Notation

Unless otherwise noted, all definitions and notation are consistent with [7]. Throughout this document a *graph* G will be a finite, nonempty set V(G) of elements called *vertices* and a (possibly empty) set E(G) of unordered pairs of distinct vertices of V(G) called *edges*. Vertices are represented by single lower case letters, possibly subscripted, such as u, w, or v_2. Edges will be denoted by either the letter e

appropriately subscripted or superscripted, or by listing the two vertices which are its endpoints. For example, $e = uw$ means that e is an edge between the vertices u and w. All graphs considered in this study are assumed to be graphs without self-loops ($uu \notin E(G)$) or multiple edges (the edge uw does not appear twice in E(G)). If $e = uw \in E(G)$ then u and w are *adjacent* vertices while e is *incident* with u and w.

The degree of a vertex $v \in V(G)$, denoted by $\deg_G(v)$ or $\deg(v)$ if G is clear from the context, is the number of edges of G incident with v. The trivial graph consists of one isolated vertex where an *isolated* vertex is a vertex of degree zero. In our study all vertices $v \in V(G)$ have degree of at least one. A vertex of degree one is called an *endvertex* and its corresponding edge is a *pendant* edge. The minimum degree of a vertex in G is denoted by $\delta(G)$ while a graph G is *regular (of degree r)* if for each vertex $v \in V(G)$, $\deg(v) = r$, for integer $r \geq 0$. A graph (or subgraph) is *complete* if every two of its vertices are adjacent.

If S_v is a set of vertices of G we use $G - S_v$ for the graph obtained from G by deleting all vertices of S_v with their incident edges. If S_e is a set of edges of G, then $G - S_e$ is the graph on the same vertex set as G with edge set $E(G) - S_e$. For an edge $e \notin E(G)$ whose incident vertices are u and w, we use $G + e$ or $G + uw$ to denote the graph whose vertex set is V(G) and whose edge set is $E(G) \cup \{e\}$. If S_m is a set of graph elements of G, then $G - S_m$ is the graph obtained from G by deleting all the edges of S_m and by deleting all the vertices of S_m with their incident edges. $N_G(v)$ is the neighborhood of v, the set of all vertices adjacent to v. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If S is a set of vertices of G, then $< S >_G$ represents the *subgraph induced by S in G*, that is, the vertex set S and edge set consisting of the edges of G incident with two vertices in S.

Now we include a few of the definitions that relate some basic terms to the connectivity of a graph. If $u, v \in V(G)$ (not necessarily distinct) then a *u-v walk* is a sequence of vertices of G, beginning with u and ending with v, such that there exists

an edge between each pair of consecutive vertices in the sequence. A *u-v path* is a u-v walk in which no vertex is repeated. The number of edges in a walk or path is called its *length*. A u-v walk in which no vertex is repeated except the first and last (u = v) is called a *cycle*. A vertex u is said to be *connected* to a vertex v in a graph G if there exists a u-v path in G. A graph G is *connected* if every distinct pair of vertices of G are connected.

A *component* of a graph is a maximal (with respect to edges) connected subgraph. A cutvertex of G is a vertex whose deletion either increases the number of components or increases the number of isolates in G. Note that this definition permits either end of a K_2-component to be a cutvertex which is a slight variation of the definition of a cutvertex in standard use. This variation is essential in allowing a meaningful definition for one of the inclusive connectivity parameters. Like a cutvertex, an edge whose removal increases the number of components of the graph is called a *bridge*.

The *(vertex) connectivity* $\kappa(G)$ *of G* is the minimum number of vertices whose removal (along with associated edges) results in a disconnected or trivial graph while the *edge connectivity* $\lambda(G)$ is the minimum number of edges whose removal yields a disconnected or trivial graph. A connected graph G has $\kappa(G) \geq 1$ and $\lambda(G) \geq 1$, while a graph G has a cutvertex if and only if $\kappa(G) = 1$ and a graph has a bridge if and only if $\lambda(G) = 1$. Given any $n \geq 1$, if $\kappa(G) \geq n$ the graph G is said to be n-connected while if $\lambda(G) \geq n$ it is n-edge connected.

The graph shown in Figure 2.1 demonstrates the concepts of cutvertex and bridge where $\kappa(G) = 1$, with vertex u as a cutvertex and $\lambda(G) = 1$, with edge e as a bridge. The subgraph induced by the vertices labeled v, w, x, and y in Figure 2.1 is shown in Figure 2.2.

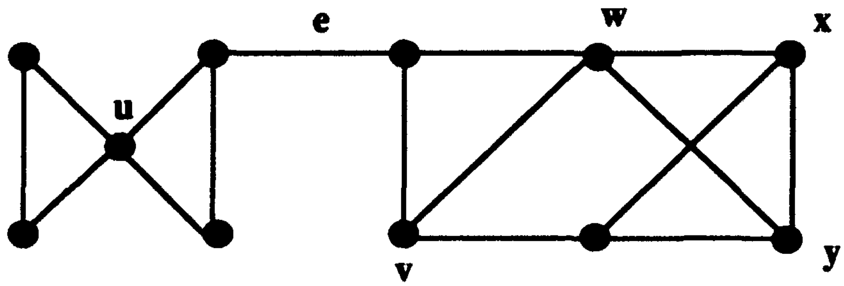


Figure 2.1 A graph with $\kappa(G) = 1$ and $\lambda(G) = 1$.

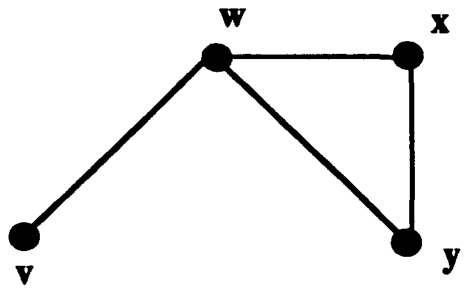


Figure 2.2 An induced subgraph of the graph in Figure 2.1.

For $v \in V(G)$, the *inclusive edge connectivity of v*, $\lambda_i(v, G)$, (formerly called cohesion in [14, 18, 22]), is the minimum number of edges whose removal yields a subgraph in which v is a cutvertex. Similarly, for $v \in V(G)$, the *inclusive vertex connectivity of v*, $\kappa_i(v, G)$, is the minimum number of vertices whose removal yields a subgraph in which v is a cutvertex and for $v \in V(G)$, the *inclusive mixed connectivity of v*, $\mu_i(v, G)$, is the minimum number of graph elements (vertices and edges) whose removal yields a subgraph in which v is a cutvertex.

For $e \in E(G)$, the inclusive connectivity parameters for edges $\lambda_i(e, G)$, $\kappa_i(e, G)$, $\mu_i(e, G)$ are defined similarly where "cutvertex" is replaced by "bridge" in the preceding definitions.

Thus $v \in V(G)$ is a cutvertex in G if and only if $\lambda_i(v, G) = \kappa_i(v, G) = \mu_i(v, G) = 0$ and $e \in E(G)$ is a bridge in G if and only if $\lambda_i(e, G) = \kappa_i(e, G) = \mu_i(e, G) = 0$.

When the underlying graph is apparent, reference to that graph may be suppressed, for instance we may use $\lambda_i(v)$ instead of $\lambda_i(v, G)$ when no confusion arises. Inclusive connectivity is also referred to as i-connectivity.

If S is a smallest set of edges (respectively vertices, graph elements) whose removal from G makes v a cutvertex, then we call S a λ_i-set (respectively κ_i-set, μ_i-set) for v in G. If S is a λ_i-set (respectively κ_i-set, μ_i-set) for v in G and G - v - S has a neighbor of v as an isolated vertex then we say that S is a neighborhood λ_i-set (respectively κ_i-set, μ_i-set) for v in G. In a complete graph, every λ_i, κ_i and μ_i-set is respectively a neighborhood λ_i, κ_i, and μ_i-set.

The graph in Figure 2.3 illustrates these parameters. Here $\lambda_i(v, G) = 2$, $\kappa_i(v, G) = 3$, and $\mu_i(v, G) = 2$. There are several μ_i-sets for v, {bc, uw}, {b, uw}, or {c, uw}, for example, but there is only one λ_i-set for v, {bc, uw}. Note that the only κ_i-sets for v in this graph involve neighborhood κ_i-sets; namely {a, b, w} for neighbor u of v and {u, c, d} for w, the other neighbor of v. Both possible κ_i-sets reduce v to part of a K_2 component in $G - S_v$, where S_v = {a, b, w} or {u, c, d}. The reader can verify that a vertex v will have only neighborhood κ_i-sets whenever $< N_G(v) >$ is complete. It should be noted that the existence of only neighborhood κ_i-sets for v does not imply $< N_G(v) >$ is complete. For example every vertex in the graph $G = C_4$, the cycle of four vertices, has only one κ_i-set, which happens to be a neighborhood κ_i-set, but no neighborhood of any vertex is complete.

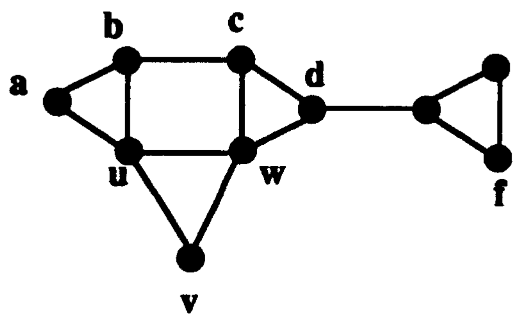


Figure 2.3 A graph illustrating the inclusive connectivity parameters.

Using neighborhood sets for possible λ_i, κ_i, and μ_i-sets, we can quickly obtain an upper bound for all three i-connectivity parameters.

Theorem 2.1: Given any graph G and any $v \in V(G)$,

$$\max \{ \lambda_i(v, G), \kappa_i(v, G), \mu_i(v, G) \} \leq \min \{ \deg_G(w) : w \in N_G(v) \} - 1.$$

There are alternative definitions for the inclusive connectivity parameters that are extremely useful in applications. Using minimum separating sets for the subgraph induced by the neighborhood of a vertex v in G - v, will be a frequently used method of examining inclusive connectivity parameters. This method was first established in the next result by Lipman and Ringeisen from [14].

Theorem 2.2: For any graph G and $v \in V(G)$, if $\deg_G(v) \geq 2$ then $\lambda_i(v)$ is the size of the smallest set of edges whose removal from G - v separates vertices of N(v) into different components.

In the same manner Boland [2] established the following for the κ_i and μ_i parameters.

Theorem 2.3: Given any graph G and $v \in V(G)$, if $\deg_G(v) \geq 2$ then $\kappa_i(v)$ is the size of the smallest set of vertices whose removal from G - v either separates two vertices from N(v) into different components or isolates a neighbor of v.

Theorem 2.4: Given any graph G and $v \in V(G)$, if $\deg(v) \geq 2$ then $\mu_i(v)$ is the size of the smallest set of graph elements whose removal from G - v separates vertices of N(v) into different components.

Theorem 2.3 concerning the κ_i parameter is slightly different from Theorems 2.2 and 2.4 since adjacent vertices can never be separated into different components by removing vertices. For the other parameters, it is always possible to separate two neighbors of v (adjacent or non-adjacent) into different components.

The only situation left unresolved from the previous theorems is the one where v is a vertex of degree one and thus there is no pair of neighbors to separate. In this case we are limited to taking a neighborhood set of the only neighbor, obtaining $\lambda_i(v, G) = \kappa_i(v, G) = \mu_i(v, G) = \deg_{G-v}(u)$ where u is adjacent to v.

The graph in Figure 2.4 illustrates that the neighborhood of v being complete is not necessary for v to be part of a K_2 component in G - S_v, for any κ_i-set S_v.

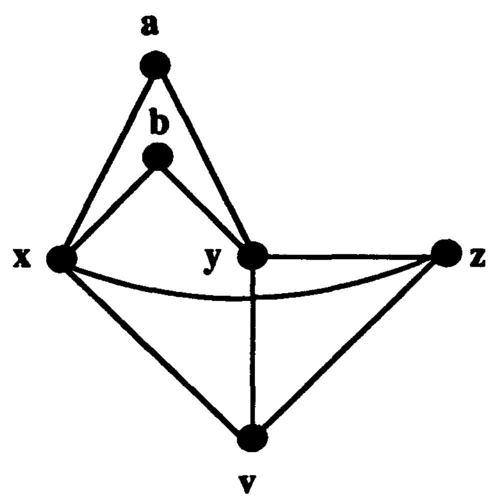


Figure 2.4 A graph illustrating the separation of neighbors of v.

There is only one κ_i-set for v, namely {x, y}, while v is part of a K_2 component in G - x - y, even though < $N_G(v)$ > is not complete. A μ_i-set for v in G is {yz, x} which separates the neighbors y and z in G - v - yz - x, and the only λ_i-set for v in G is {yz, xz}, which separates the neighbors x and y from z in G - v -yz - xz.

A *block* of a graph is a maximal induced subgraph which contains no cutvertex (in the usual sense of a cutvertex). Rice first observed [18] that blocks of a graph play an important role in determining the inclusive connectivity parameters. The following results were expanded to include all three parameters in [3].

Theorem 2.5: Given any graph G and v ∈ V(G), the elements of any λ_i, κ_i, or μ_i-set for v are contained in a single block of G.

Theorem 2.6: Given any graph G and e ∈ E(G), the elements of any λ_i, κ_i, or μ_i-set for e are contained in a single block of G.

In particular these results allow us to only examine components for i-connectivity parameters, hence we assume throughout this dissertation that G is a connected graph.

All the elements of i-connectivity sets of vertex v ∈ V(G) in the graph in Figure 2.5 are contained in the block consisting of {v, a, b, c, d}. Specifically, the λ_i-set consists of {cd}, the κ_i-set consists of {c}, and a μ_i-set consists of either {cd} or {c}.

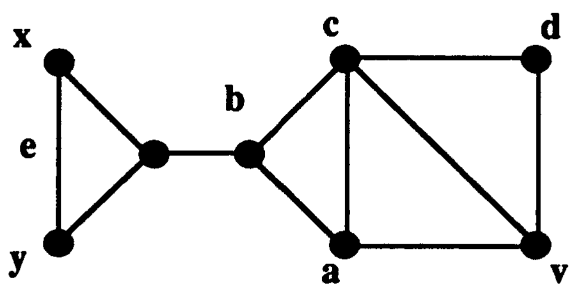


Figure 2.5 A graph illustrating Theorem 2.5.

The operation of ***subdividing an edge*** e = xy consists of replacing the edge e with a pair of edges xz and zy where z $\notin$ V(G). If the edge e in Figure 2.5 is subdivided, then we get the graph in Figure 2.6. The operation of subdividing an edge is important in the study of inclusive connectivity for edges which will be discussed later in this chapter.

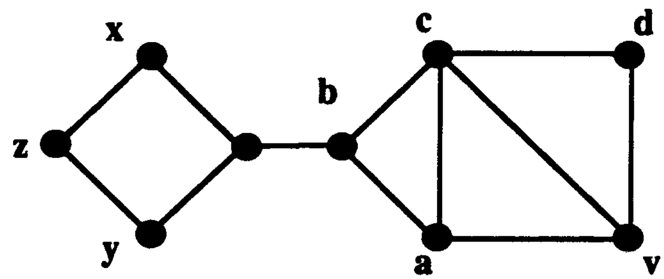


Figure 2.6 A graph of Figure 2.5 with edge e subdivided.

One of our primary concerns will be changes in i-connectivity values upon the addition or deletion of an edge to G; thus we define a vertex v $\in$ V(G) as *λ_i-stable under edge addition* if λ_i(v, G) = λ_i(v, G + e) for every edge e $\notin$ E(G). Similarly a vertex v $\in$ V(G) is *κ_i- (μ_i) stable under edge addition* if κ_i(v, G) = κ_i(v, G + e) (μ_i(v, G) = μ_i(v, G + e)) for every edge e $\notin$ E(G). In the graph in Figure 2.3, if e = vf then λ_i(v, G + e) = κ_i(v, G + e) = μ_i(v, G + e) = 1. Hence vertex v is not stable under edge addition. Every complete graph will be trivially stable under edge addition for all three parameters. Chapter 3 provides more examples of graphs exhibiting various types of stability under edge addition.

For edge deletion we define a vertex v $\in$ V(G) is said to be *λ_i-stable under edge deletion* if λ_i(v, G) = λ_i(v, G - e) for every edge e $\in$ E(G). A vertex v $\in$ V(G) is said to be *κ_i- (μ_i) stable under edge deletion* if κ_i(v, G) = κ_i(v, G - e) (μ_i(v, G) = μ_i(v, G -

e)) for every edge e ∈ E(G). This area of study is thoroughly investigated in Chapter 4.

Finally, we define what we term neutral edges of a graph (with respect to i-connectivity). An edge e ∈ E(G) is said to be *λ_i-neutral* if $\lambda_i(v, G) = \lambda_i(v, G - e)$ for all v ∈ V(G). That is, upon the deletion of edge e, the λ_i value for every vertex remains the same. This definition is identical to the "stable edge" definition in [19]. The reason for the terminology change is to insure no confusion between an edge whose deletion does not affect the i-connectivity values for any vertex of a graph and an edge whose own i-connectivity values do not change under some graph operation. In a similar manner we say an edge e ∈ E(G) is called *κ_i- (μ_i) neutral* if $\kappa_i(v, G) = \kappa_i(v, G - e)$ ($\mu_i(v, G) = \mu_i(v, G - e)$) for all v ∈ V(G). In the graph in Figure 2.7 we note that edge e is λ_i, κ_i, and μ_i-neutral since $\lambda_i(v, G) = \lambda_i(v, G - e) = \kappa_i(v, G) = \kappa_i(v, G - e) = \mu_i(v, G) = \mu_i(v, G - e) = 1$ for every v ∈ V(G). Research into inclusive connectivity neutral edges is presented in Chapter 5.

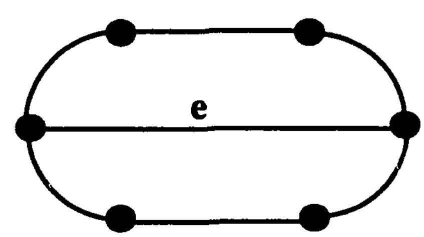


Figure 2.7 A λ_i, κ_i, and μ_i-neutral edge.

Well Known Graph Theory Results

A well known result relating a graph's edge connectivity and vertex connectivity is credited to Whitney [25].

Theorem 2.7 (Whitney): For any graph G, $\kappa(G) \le \lambda(G) \le \delta(G)$.

The graph in Figure 2.8 demonstrates that Whitney's Theorem can occur as a strict inequality with $\kappa(G) = 2$, $\lambda(G) = 3$, and $\delta(G) = 4$.

An interesting difference we between the i-connectivity parameters and their global counterparts is that there exist three different i-connectivity parameters, as opposed to two global parameters. This is because it is known that the mixed connectivity of a graph is the same as its vertex connectivity.

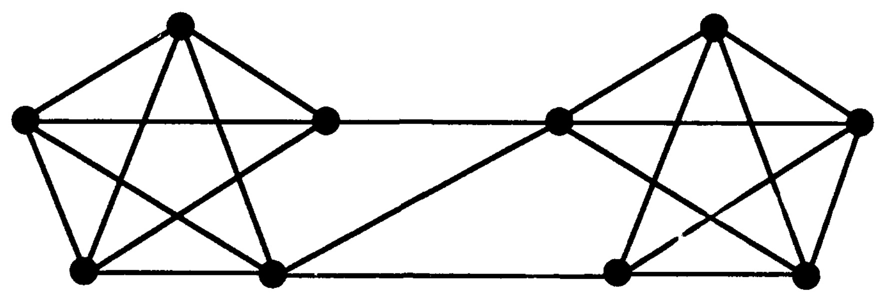


Figure 2.8 A graph illustrating Whitney's Theorem.

Another significant difference between these two sets of parameters is evident by examining Figure 2.3 again. Here, we have $\kappa_i(v) > \lambda_i(v)$, which is in stark contrast to Whitney's Theorem for the global parameters. Boland [2] conducted a thorough study of these relationships and showed that every possible relationship between the three i-connectivity parameters is attainable subject to $\mu_i(v) \le \kappa_i(v), \lambda_i(v)$.

Two other important results that we shall use extensively in our work are Menger's Theorem and the edge analog of Menger's Theorem. Both of these theorems provide us with a convenient aid in computing our parameters and establishing results, when combined with the method of "separating neighbors" previously discussed.

A set S of edges (or vertices) of a graph G is said to *separate* vertices u and v if the removal of the elements of S produces a disconnected graph in which u and v lie in

different components. Such a set S is called a *separating set or cutset* for u and v. Two u-v paths are *internally disjoint* if they have no vertices in common, other than u or v, while *edge disjoint* u-v paths have no edges in common. It is obvious that internally disjoint paths are also edge disjoint. Throughout this dissertation the term "n internally (edge) disjoint paths" means a set of n paths so that any two are internally (edge) disjoint. If $n = 1$, the set is vacuously internally (edge) disjoint.

Theorem 2.8 (Menger): Let u and w be nonadjacent vertices in a graph G. Then the minimum number of vertices that separate u and w is equal to the maximum number of internally disjoint u-w paths.

Theorem 2.9 (Menger): If u and w are distinct vertices of a graph G, then the maximum number of edge disjoint u-w paths in G equals the minimum number of edges that separate u and w.

The "separating neighbors" conceptualization of i-connectivity suggests that Menger's Theorem be used in computing the parameters. For instance, the smallest set of vertices whose removal from G - v separates vertices from N(v) into different components is the same as the maximum number of internally disjoint paths in G - v, among pairs of vertices from N(v). Algorithms for computing the i-connectivity parameters for vertices have been implemented in [12]. The i-connectivity values for all graphs given in this text were verified using that software.

Literature Review

Investigation into the structure and connectivity of graphs has always been fundamental in graph theory. Expanding upon Whitney's result [25], Chartrand and Harary [6] proved that, given integers a, b, and c, $a \le b \le c$, there exists a graph G with $\kappa(G) = a$, $\lambda(G) = b$, and $\delta(G) = c$. Chartrand [5] further characterized when $\lambda(G) = \delta(G)$, while Lesniak [13] continued work in this area.

Since a mixed connectivity and vertex connectivity parameter will always have the same value in the global sense, there had been little focus on mixed disconnecting sets until Yau in 1962 [26] considered cutsets in graphs including but not limited to (minimal) cutsets consisting of both vertices and edges.

In another approach, Beineke and Harary [1] defined the connectivity function of a graph that was based on the idea of a connectivity pair. A connectivity pair is an ordered pair of non-negative integers, (i, j), such that there is a set of i vertices and j edges whose removal disconnects the graph, and there is no set of i - 1 vertices and j edges or i vertices and j - 1 edges which also disconnects the graph upon removal. The connectivity function generated some interest, but there has been very little research on the properties of connectivity pairs.

In a related approach, Chartrand and Pippert [8] first defined "locally connected". A graph G is said to be *locally connected* if $\langle N(v) \rangle$ is connected for every $v \in V(G)$. The edge analog is similarily defined. These were the first localizations of connectivity, and were extensively studied in [11, 15, 16, 23, 24]. However, vertices from outside the neighborhood of a vertex can significantly impact how well a graph is connected in the local area around that vertex. This provides some motivation for the i-connectivity parameters because contributions from vertices outside the neighborhood are inherently accounted for.

The first inclusive connectivity parameter introduced was cohesion (our λ_i parameter) by Lipman and Ringeisen [14] in 1979 and was further expanded in [2-4]. These i-connectivity parameters are local measures of graph vulnerability. In fact, they were shown in [3] to be natural localizations of graph connectivity and edge connectivity.

Ringeisen and Lipman [22] continued their work in inclusive connectivity in 1983 by considering the effects of edge addition on cohesion and introduced the concept of vertex stability under edge addition. At the same time Reid [17] extended and

independently verified some of the results of [14] and [22] by examining when the cohesion of a vertex v is less than the edge connectivity of G - v.

In a sequence of papers, Rice and Ringeisen [19-21] explored the areas of stable edges and stable graphs. As stated previously, Rice's stable edges are identical to our definition of neutral edges. A graph G is defined to be stable if, upon the deletion of any edge of G, the sum of all the λ_i values of the vertices remained the same. This definition of a graph being stable involves the sum of all λ_i values instead of an individual λ_i value remaining the same, since the removal of any edge in a λ_i-set for vertex v will cause the λ_i value for v to decrease. Surprisingly, several infinite classes of stable graphs were found.

Following Rice, Boland and Ringeisen [2-4] extended the cohesion results to include vertex and mixed inclusive connectivity. These papers were the first to use the inclusive connectivity terminology. Boland's study included the inclusive connectivity values for certain composite graphs as well as the first look into super i-connected graphs. A graph is *super* λ *(super* κ*)* if every edge (vertex) disconnecting set of size λ (κ) isolates a vertex.

In 1990, Lee [12] implemented the first software package to compute all three i-connectivity parameters for a graph. This package has been used extensively to verify the results of this dissertation.

Inclusive connectivity was studied for extremal properties by Lai and Lai [10] who investigated the maximum and minimum number of edges a graph contained, when one fixed the minimum inclusive edge connectivity of a vertex in the graph, and the number of vertices in the graph.

Most recently, Cribb, Boland, and Ringeisen [9] extensively examined conditions for stability under edge addition, for all three inclusive connectivity parameters.

Extensions of Previous Results

Most of the cornerstone theorems regarding the effect of edge addition on inclusive edge connectivity values were established by Ringeisen and Lipman in [22]. Arguably the most important theorem in this area of study is Theorem 2.10. It precisely states the conditions under which the λ_i value for a vertex can increase or decrease after edge addition.

Theorem 2.10: Let u, v, and w be distinct vertices of graph G and edge e = uw where uw $\notin$ E(G). Then

(a) $\lambda_i(v, G) \leq \lambda_i(v, G + e) \leq \lambda_i(v, G) + 1$ and

(b) $\lambda(G - u) \leq \lambda_i(u, G + e) \leq \lambda_i(u, G)$.

Simply put, if an edge e is added to a graph G and is incident to v $\in$ V(G), then the λ_i value for v can only remain the same or decrease. This is because $N_G(v) \subset N_{G+e}(v)$, which implies any λ_i-set for v in G will still separate the same pair of neighbors in G + e - v with possible decrease in the λ_i value due to the new neighbor. On the other hand if e is not incident to v, then the λ_i value for v can only remain the same or increase by exactly one. In this case if S is a λ_i-set for v in G, then S can possibly remain a λ_i-set for v in G + e or at worst the removal of S $\cup$ {e} from G + e - v will separate the same pair of neighbors.

Boland [2] extended this result to include mixed inclusive connectivity in Theorem 2.11.

Theorem 2.11: Let u, v, and w be distinct vertices of graph G and edge e = uw where uw $\notin$ E(G). Then

(a) $\mu_i(v, G) \leq \mu_i(v, G + e) \leq \mu_i(v, G) + 1$ and

(b) $\kappa(G - u) \leq \mu_i(u, G + e) \leq \mu_i(u, G)$.

We can also extend these results to include the κ_i-parameter, but here we will begin to realize some of the differences in this parameter. Possibly our first such discovery was during the discussion of Whitney's Theorem in relation to inclusive

connectivity. Intuitively we can think of a vertex as "behaving normally" if its i-connectivity responds in a manner paralleling Whitney's Theorem (i.e. $\kappa_i(v) \leq \lambda_i(v)$). But as discussed previously, this is not true in every case since we have seen in the graph in Figure 2.8 a case where $\kappa_i(v) > \lambda_i(v)$. In the previous two theorems, we can see that when the λ_i and μ_i increase, they can increase by at most one. As Theorem 2.12 states, this is not true for κ_i.

Theorem 2.12: Let u, v, and w be distinct vertices of graph G and edge e = uw where uw $\notin$ E(G). Then

(a) $\kappa_i(v, G) \leq \kappa_i(v, G + e)$ and

(b) $\kappa(G - u) \leq \kappa_i(u, G + e) \leq \kappa_i(u, G)$.

Proof: Let S_v be a κ_i-set for v in G + e. Then v is a cutvertex in (G + e) - S_v. But v would remain a cutvertex in G - S_v since $N_G(v) = N_{G+e}(v)$ implies the same pair of neighbors of v would be separated in G - v - S_v. So $\kappa_i(v,G) \leq | S_v | = \kappa_i(v, G + e)$.

To establish (b), notice that since (G + e) - u = G - u, any κ_i-set for u in G will separate the same pair of neighbors in G + e - u since $N_G(u) \subset N_{G+e}(u)$. So $\kappa_i(u, G + e) \leq \kappa_i(u, G)$.

Finally, if S_u is a κ_i-set for u in G + e, then S_u is a disconnecting set of vertices for G - u, since by definition (G + e - u) - S_u = (G - u) - S_u is either disconnected or the trivial graph. Thus $\kappa(G - u) \leq \kappa_i(v, G + e)$. ❑

An example of when κ_i increases by more than one is illustrated in the graph in Figure 2.9. There $\kappa_i(v, G) = 5$ with the separation of neighbors u and w of v, but after the addition of edge e = uw, $\kappa_i(v, G + e) = 8$ with the separation of neighbors x and y. The reason that the κ_i parameter increases in this manner is that it becomes impossible to separate the adjacent neighbors u and w in G + e by deleting vertices alone.

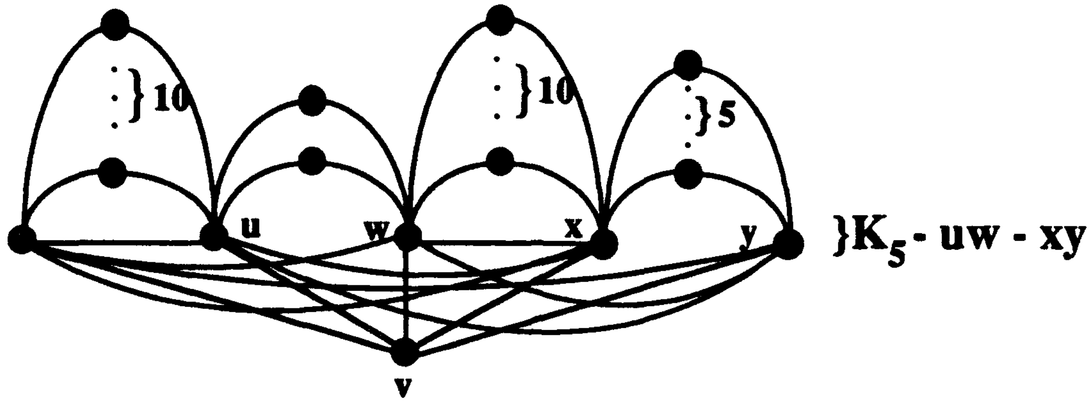


Figure 2.9 The increase in the κ_i parameter.

We can easily show that the increase in κ_i can be arbitrarily large. If an increase of n, $n \geq 1$, is desired, then we simply change the two columns of 10 vertices of degree two to two columns of 7 + n vertices of degree two and the one column of 5 vertices of degree two to a column of 2 + n vertices of degree two.

Ringeisen and Lipman also discovered a characterization of exactly when the edge i-connectivity of a vertex ($\lambda_i(v)$) increases, and this was followed by a similar partial result by Boland for mixed i-connectivity.

Theorem 2.13: Let u, v, and w be distinct vertices of G with uw $\notin$ E(G) and $\lambda_i(v,G) > 0$. Then $\lambda_i(v, G + uw) > \lambda_i(v, G)$ if and only if uw is a bridge in $(G + uw) - v - S_e$ for every S_e which is a λ_i-set for v in G.

Theorem 2.14: Let u, v, and w be distinct vertices of G with uw $\notin$ E(G). If $\mu_i(v, G + uw) > \mu_i(v, G)$ then uw is a bridge in $(G + uw) - v - S_m$ for every S_m which is a μ_i-set for v in G.

A similar partial result for vertex i-connectivity is now presented.

Theorem 2.15: Let u, v, and w be distinct vertices of G with uw $\notin$ E(G). If $\kappa_i(v, G + uw) > \kappa_i(v, G)$ then uw is a bridge in $(G + uw) - v - S_v$ for every S_v which is a κ_i-set for v in G.

Proof: Suppose $\kappa_i(v, G + uw) > \kappa_i(v, G)$ and let S_v be any κ_i-set (possibly empty) for v in G. Now by definition v is a cutvertex in $G - S_v$ but it is not in $G + uw - S_v$ since $\kappa_i(v, G + uw) > \kappa_i(v, G) = |S_v|$. So $G - v - S_v$ is disconnected with a greater number of components than $G + uw - v - S_v$ and since $(G + uw) - v - (S_v \cup \{uw\}) = G - v - S_v$, then uw is a bridge in $(G + uw) - v - S_v$. ❑

It should be noted that it was not possible to establish a likewise characterization for the κ_i and μ_i parameters. The graph in Figure 2.10 demonstrates that the contrapositives of Theorems 2.14 and 2.15 are not valid by displaying a vertex v with $\kappa_i(v, G) = \mu_i(v, G) = 1$ which has as its only κ_i or μ_i- set the vertex y.

Upon examination it can be seen that $\kappa_i(v, G + uw) = \mu_i(v, G + uw) = 1$ where the κ_i and μ_i-sets still consist of just the vertex y. But uw is a bridge in $(G + uw) - v - S$ for every S which is a κ_i or μ_i-set for v in G.

Even though the previous extensions of the Ringeisen and Lipman results [22] for λ_i were not identical for κ_i and μ_i, the following propositions show that some results for κ_i and μ_i are completely analogous under the operation of edge addition.

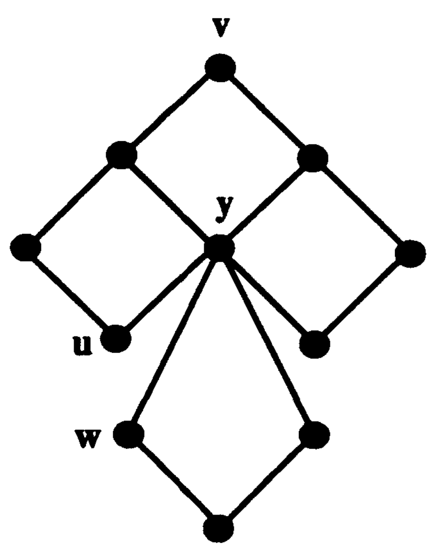


Figure 2.10 A counterexample to the converses of Theorems 2.14 and 2.15.

Ringeisen and Lipman's work provided some examples of graphs that were λ_i-stable under edge addition, i.e. every vertex in the graph was λ_i-stable under edge addition. The first result toward this goal is Proposition 2.16.

Proposition 2.16: If $v \in V(G)$ is adjacent to at least three distinct vertices each of which has degree at most $\lambda(G - v) + 1$, then v is λ_i-stable under edge addition.

An analogous result for κ_i and μ_i is now presented.

Proposition 2.17: If $v \in V(G)$ is adjacent to at least three distinct vertices each of which has degree at most $\kappa(G - v) + 1$, then v is κ_i and μ_i-stable under edge addition.

Proof: This proof will establish κ_i-stability under edge addition since the argument for μ_i-stability is identical.

Let $u \in N_G(v)$ where u is one of the three distinct vertices that has degree at most $\kappa(G - v) + 1$ and deg(u) is the degree of u in G. Then by Theorem 2.1, $\kappa_i(v, G) \leq \deg(u) - 1$. Since it is given that $\deg(u) \leq \kappa(G - v) + 1$ we have $\kappa_i(v, G) \leq \kappa(G - v)$. And since $\kappa_i(v, G) \geq \kappa(G - v)$ is always true, then we have equality.

If G is a complete graph, then v is κ_i-stable under edge addition by definition. Assume G is not complete and let x and y be nonadjacent vertices of G. Since $\kappa((G + xy) - v) \geq \kappa(G - v)$ and $\kappa_i(v, G + xy) \geq \kappa(G + xy - v)$, we have $\kappa_i(v, G + xy) \geq \kappa(G - v)$. By the hypothesis, there is a $w \in N_G(v)$ where w is not x or y and the degree of w in G + xy is at most $\kappa(G - v) + 1$. Arguing as before, $\kappa_i(v, G + xy) \leq \kappa(G - v)$. Combining results produces $\kappa_i(v, G + xy) = \kappa(G - v) = \kappa_i(v, G)$ which implies κ_i-stability under edge addition. ❑

Two corollaries follow that provide the desired examples for all the parameters. The proofs follow that used for λ_i in [22] and have been omitted.

Corollary 2.18: If G is regular of degree r, $r \geq 3$, and $v \in V(G)$ with $\lambda(G - v) = r - 1$ ($\kappa(G - v) = r - 1$), then v is λ_i- (κ_i and μ_i) stable under edge addition.

Corollary 2.19: The following graphs are λ_i, κ_i, and μ_i-stable under edge addition.

(a) The Petersen Graph

(b) The complete bipartite graphs K(n, n), n ≥ 3.

Even though the major thrust of this work is not on graphs which are i-connectivity stable under edge addition, it is interesting to note that there do exist several nontrivial infinite classes of such graphs.

Inclusive Connectivity for Edges

We present a preliminary investigation of i-connectivity for edges. Most previous work regarding i-connectivity dealt primarily with vertices. We now examine the relationships between the i-connectivity parameters defined for edges showing that the previously established results for vertices are useful in establishing parallel results for edges. For any $e = uw \in E(G)$ let G^* be the graph G with edge e subdivided. We label the vertex introduced upon subdivision as v. Proposition 2.23 presents the basic idea in this investigation.

Proposition 2.20: An edge e is a bridge in G if and only if v is a cutvertex in G^*.

Proof: Let e be a bridge in G with endpoints u and w. Then G - e consists of exactly two components with u and w being in different components. Then, in G^*, v is adjacent to both u and w implying G^* is connected with v being on every u-w path of G^*. Thus v is a cutvertex in G^*. The converse argument is similar. ❑

Figure 2.11 depicts this fundamental idea of viewing any bridge as a cutvertex of degree two in a slightly modified graph.

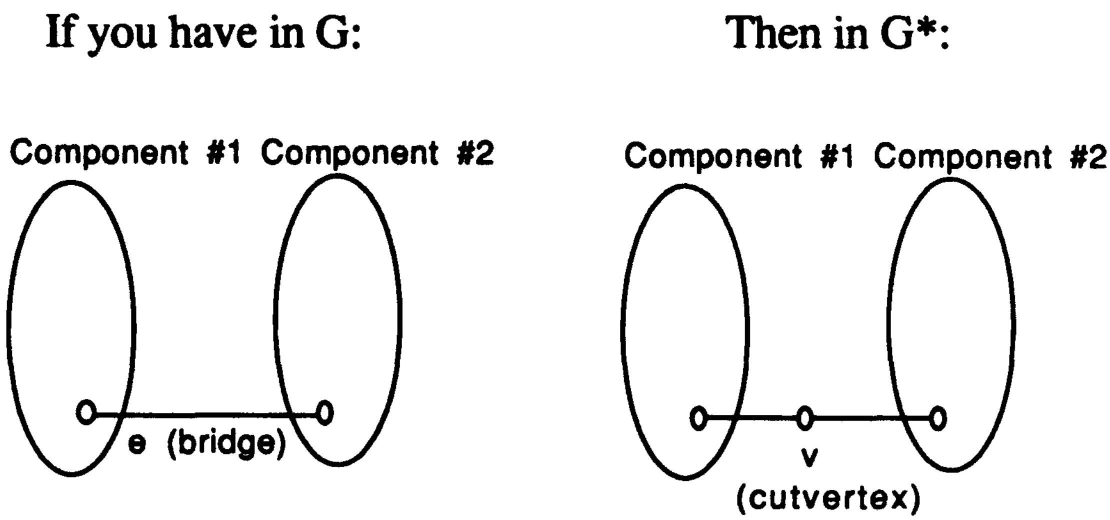


Figure 2.11 Inclusive connectivity for edges.

Corollary 2.21: Given a graph G' and $v \in V(G')$, if v has degree two and its neighbors u and w are not adjacent in G', then $\lambda_i(v, G') = \lambda_i(e, G)$ where e = uw and G = G' - v + uw.

Proof: The reverse of the subdivide operation produces G' - v + uw = G. Thus G' - v = G - e. Let S_e' be any λ_i-set for v in G'. Then $G' - v - S_e' = G - e - S_e'$ which implies e is a bridge in $G - S_e'$. Therefore $\lambda_i(v, G') \geq \lambda_i(e, G)$. Assume there exists a λ_i-set S_e for e in G such that $| S_e | < | S_e' |$. Then $G' - v - S_e = G - e - S_e$ which implies v is a cutvertex in $G' - S_e$ and we have $| S_e | \geq \lambda_i(v, G') > \lambda_i(e, G) = | S_e |$, a contradiction. Therefore $\lambda_i(v, G') = \lambda_i(e, G)$. ❑

Two additional corollaries which follow show that identical results hold for the remaining two i-connectivity parameters. The proofs are identical to those of Corollary 2.21 and have been omitted.

Corollary 2.22: Given a graph G' and $v \in V(G')$, if v has degree two and its neighbors u and w are not adjacent in G', then $\kappa_i(v, G') = \kappa_i(e, G)$ where e = uw and G = G' - v + uw.

Corollary 2.23: Given a graph G' and $v \in V(G')$, if v has degree two and its neighbors u and w are not adjacent in G', then $\mu_i(v, G') = \mu_i(e, G)$ where $e = uw$ and $G = G' - v + uw$.

Proposition 2.20 and Corollaries 2.21-2.23 indicate that many results on inclusive connectivity for vertices in [2] hold for edges, considering the subdivide operation.

Next is a characterization of when an edge is in a minimum disconnecting set for a graph in terms of its inclusive edge connectivity from [2].

Proposition 2.24: Given a graph G, and $e \in E(G)$,

$\lambda_i(e) = \lambda(G) - 1$ if and only if e is in a minimum edge disconnecting set of G.

Proof: Let $\lambda_i(e) = \lambda(G) - 1$ and let S_e be any λ_i-set for e in G. So $| S_e | = \lambda_i(e)$ and $\lambda_i(e) + 1 = \lambda(G)$. Since $G - e - S_e$ is disconnected (by the definition of S_e), and $| S_e \cup \{e\} | = | S_e | + 1 = \lambda(G)$, then e is in a minimum edge disconnecting set of G.

Let e be in a minimum edge disconnecting set, S_e, of G. Then $\lambda(G) = | S_e |$ and $G - (S_e - e) = G + e - S_e$ is connected by the minimality of S_e. So $\lambda_i(e) \le | S_e | - 1 = \lambda(G) - 1$. But we also know $\lambda(G) = \min \{\lambda_i(e) : e \in E(G)\} + 1$. So $\lambda(G) \le \lambda_i(e) + 1$ for all $e \in E(G)$ which implies $\lambda(G) - 1 = \lambda_i(e)$. ❑

The following result shows that the realizable relationships between the i-connectivity parameters defined for an edge are much more limited than those for a vertex described in [4].

Theorem 2.25: For any graph G and any edge $e = uw \in E(G)$,

$$\mu_i(e) = \kappa_i(e) \le \lambda_i(e) \le \min \{ \deg(u), \deg(w) \} - 1.$$

Proof: From the definitions of the inclusive connectivity parameters, we have $\mu_i(e) \le \kappa_i(e)$ for any $e \in E(G)$. Suppose there exists an edge $e \in E(G)$ where $\mu_i(e) < \kappa_i(e)$. Let S_v be a κ_i-set for e in G and S_m be a μ_i-set for e in G. If there are no edges in S_m, then $| S_m | < | S_v |$ contradicts the minimality of S_v. So there must be at least one edge in S_m. Construct a set of vertices $S_v{*}$ by including all the vertices in S_m, and for each edge in S_m, select one of its incident vertices that is not incident with e to be in

S_v^*. Then e will be a bridge in G - S_v^* where S_v^* consists entirely of vertices and $|S_v^*| \leq |S_m|$. Thus $\kappa_i(e) \leq |S_v^*| \leq |S_m| < |S_v| = \kappa_i(e)$ a contradiction. Therefore $\mu_i(e) = \kappa_i(e)$.

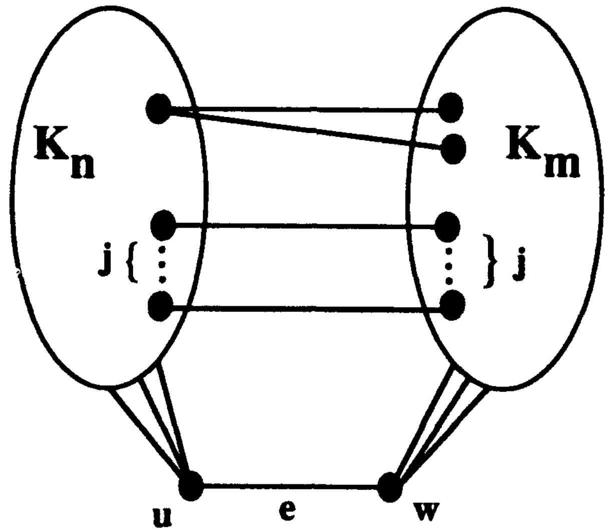
Let $e = uw \in E(G)$ and let S_e be a λ_i-set for e in G. Then e is a bridge in G - S_e and G - e - S_e consists of exactly two components. For each edge in S_e, we select exactly one of its incident vertices that is not u or w, to form a set of vertices S_v^*. Now e will be a bridge in G - S_v^* where G - e - S_v^* consists of at least two components. Thus $\kappa_i(e) \leq \lambda_i(e)$.

The last inequality is clear. ❑

Thus, when considering the relationships between the inclusive connectivity parameters for edges, there are only two possible combinations as opposed to the six for vertices shown in [4].

Notice that Theorem 2.25 indicates that the i-connectivity parameters, when defined for an edge, behave in a manner analogous to the global connectivity parameters, i.e., Theorem 2.25 is much like Whitney's Theorem. The equality $\mu_i(e,G) = \kappa_i(e, G)$ is mirrored by the previous statement that the minimum number of graph elements whose removal will disconnect G is equal to the vertex connectivity of G.

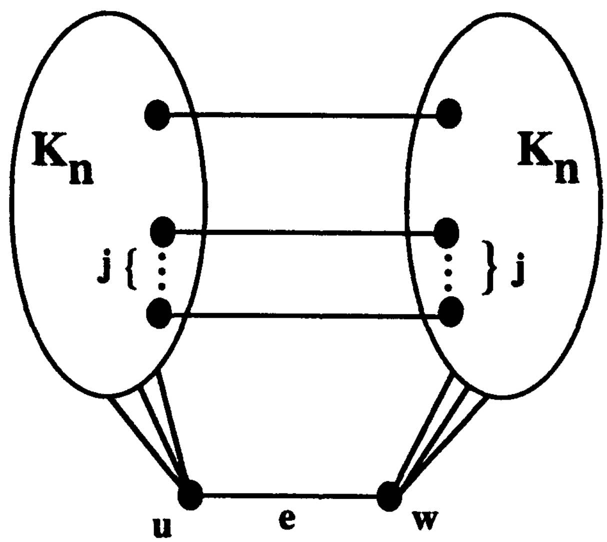
Graphs illustrating the two possible relationships between the three inclusive connectivity parameters for edges are shown in Figures 2.12 and 2.13. For the graph in Figure 2.12, $\lambda_i(e) = j + 2$ and $\kappa_i(e) = j + 1$ for $j < n \leq m$ where j, m, and n are positive integers. However, for the graph in Figure 2.13 $\lambda_i(e) = \kappa_i(e) = j+1$, $j < n$.



deg(u) = n + 1   deg(w) = m + 1

$j < n \leq m$

Figure 2.12 $\mu_i(e) = \kappa_i(e) < \lambda_i(e)$.



deg(u) = n + 1   deg(w) = n + 1

$j < n$

Figure 2.13 $\mu_i(e) = \kappa_i(e) = \lambda_i(e)$.

Both of the above figures represent an infinite class of graphs, but the reader can verify that with minor adjustments the difference between the λ_i and κ_i parameters can be arbitrarily large.

When considering graph operations such as edge addition for edge i-connectivity, again, the behavior of the edges is much more simplified than its vertex counterpart. We can see from the next theorem that the only possible outcomes are that the parameters remain the same or increase by one.

Theorem 2.26: For any graph G let $e \in E(G)$. Then for any edge $e' \notin E(G)$,

(a) $\lambda_i(e, G) \leq \lambda_i(e, G + e') \leq \lambda_i(e, G) + 1$

(b) $\kappa_i(e, G) \leq \kappa_i(e, G + e') \leq \kappa_i(e, G) + 1$

(c) $\mu_i(e, G) \leq \mu_i(e, G + e') \leq \mu_i(e, G) + 1$.

Proof: Let $e \in E(G)$ and $e' \notin E(G)$.

For (a), first let S_e be any λ_i-set for e in G. Then $\lambda_i(e, G + e') \leq | S_e \cup \{e'\} | = | S_e | + 1 = \lambda_i(e, G) + 1$ since $G - S_e = (G + e') - (S_e \cup \{e'\})$.

Now let S_e^* be any λ_i-set for e in G + e'. If $e' \in S_e^*$ then $G + e' - S_e^* = G - (S_e^* - e')$ which implies $\lambda_i(e, G) \leq | S_e^* | - 1 = \lambda_i(e, G + e') - 1$ or $\lambda_i(e, G) < \lambda_i(e, G + e')$. If $e' \notin S_e^*$ then $G + e' - S_e^* - e$ consists of exactly two components of which e' is contained in one of these components. Thus the endpoints of e are in different components in $G - S_e^* - e$ which implies e is a bridge in $G - S_e^*$. Therefore, $\lambda_i(e, G) \leq | S_e^* | = \lambda_i(e, G + e')$.

For (b), let e' have endpoints u and w where, without loss of generality, u is not an endpoint of e, and let S_v be any κ_i-set for e in G. If $u \in S_v$ then $G + e' - S_v = G - S_v$ and $\kappa_i(e, G + e') \leq \kappa_i(e, G)$. If $u \notin S_v$ then $\kappa_i(e, G + e') \leq | S_v \cup \{u\} | = | S_v | + 1 = \kappa_i(e, G) + 1$ since u will be contained in one of the components of $G - S_v - e$ and the endpoints of e will remain in different components of $G + e' - (S_v \cup \{u\}) - e$.

Let $S_v{}^*$ be any κ_i-set for e in G + e'. By definition, the endpoints of e are in different components of G + e' - $S_v{}^*$ - e which will remain the case in G - $S_v{}^*$ - e. This implies e is a bridge in G - $S_v{}^*$ so $\kappa_i(e, G) \leq | S_v{}^* | = \kappa_i(e, G + e')$.

For (c), the proof is similar to (b). ❑

CHAPTER 3

STABILITY OF INCLUSIVE CONNECTIVITY UNDER EDGE ADDITION

Introduction

This chapter will begin by investigating the various possible relationships among the inclusive connectivity parameters concerning vertex stability under edge addition. Examples for each possible relationship will include an infinite class of such graphs.

Further, we are interested in any implications or dependencies that may exist when a vertex is known to have some type of i-connectivity stability. An interesting result relating κ_i and μ_i-stability under edge addition will be presented. This chapter will also answer a question about the κ_i parameter initially started in [2] regarding the situation when $\lambda_i(v) < \kappa_i(v)$.

Finally, we explore a surprising relationship between the stability of inclusive connectivity and the stability of the global connectivities under edge addition. Throughout this chapter "stable" (or "stability") will mean "stable (stability) under edge addition".

Relationships Achievable under Edge Addition

As stated in the previous chapter, Menger's Theorem and its edge analog provide us with a convenient way of viewing i-connectivity using the separation of the neighbors of a vertex. We will now present the first of these essential theorems.

For any $v \in V(G)$, let u and w be neighbors of v. Let $p_e(u,w)$ denote the maximum number of edge disjoint paths between u and w in G - v and $p_v(u,w)$ be defined as the maximum number of internally disjoint u-w paths in G - v, i.e., paths with no vertices or edges in common.

Lipman and Ringeisen [14] proved the first such result for λ_i where $\deg(v) \geq 2$, $v \in V(G)$.

Theorem 3.1: For any graph G with $v \in V(G)$,

$$\lambda_i(v) = \min \{ p_e(u,w) : u, w \in N(v) \}.$$

Boland [2] expanded this result to include μ_i by using the same principle, except with internally disjoint paths.

Theorem 3.2: For any graph G with $v \in V(G)$ having degree greater than one,

$$\mu_i(v) = \min \{ p_v(u,w) : u, w \in N(v) \}.$$

Again, the impossibility of separating adjacent neighbors by just removing vertices results in a slightly different interpretation for κ_i [2].

Theorem 3.3: For any graph G with $v \in V(G)$ having degree greater than one and $\langle N(v) \rangle$ not complete,

$$\kappa_i(v) = \min \{ p_v(u,w) : u, w \in N(v), uw \notin E(G) \}.$$

Even though the previous three theorems all are restricted to vertices of degree two or greater, this is actually no obstacle since a vertex v, of degree one has $\lambda_i(v) = \kappa_i(v) = \mu_i(v) = \deg(u) - 1$ where u is the lone neighbor of v. And if N(v) is complete, an i-connectivity set will consist of the other neighbors of a minimum degree vertex adjacent to v.

In fact, while regarding stability under edge addition (where the inclusive connectivity for a vertex remains the same upon the addition of any edge), the case of degree one vertices will be completely analyzed later in this chapter.

Now we explore whether every possible combination of stabilities among the three i-connectivities is realizable. We currently have one unresolved case; specifically the case where a vertex v is μ_i and λ_i-stable but not κ_i-stable. We will call this Case X. We will prove later that it is impossible to have κ_i-stability without μ_i-stability, which we will call Case Y. Hence, we prove Theorem 3.4, which omits these cases.

Let G_1 and G_2 be two graphs with disjoint vertex sets. The *join* G of the two graphs G_1 and G_2, denoted as $G = G_1 + G_2$, has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Theorem 3.4: Each of the five relationships of stability among the inclusive connectivity parameters, not related to **X** or **Y**, has an infinite class of graphs satisfying it.

Proof:

Case (1): A vertex that is λ_i, κ_i, and μ_i-stable.

Any vertex from a complete graph is trivially stable for all three parameters. But following the work in [22] we arrive at Corollary 2.19. This provides us with an infinite class of graphs that not only contain a vertex that is stable for all three parameters, but is such that every vertex of the graph is stable for every i-connectivity parameter. Thus no matter what vertex we choose in the graph in Figure 3.1, we have stability for all three parameters.

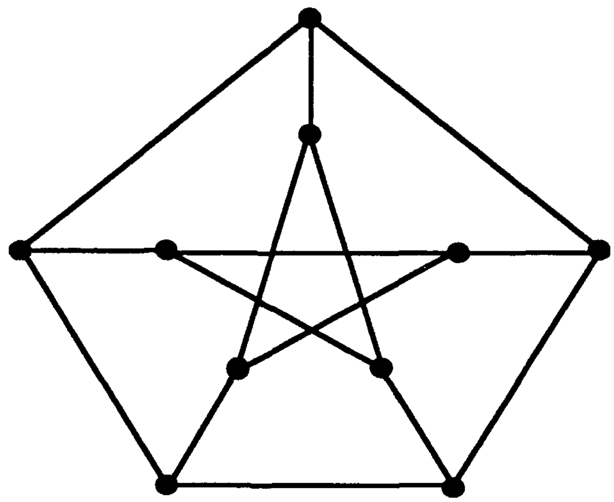


Figure 3.1 The Petersen Graph.

Case (2): A vertex that is not λ_i, κ_i, or μ_i-stable.

If v is a cutvertex in G and e $\notin$ E(G) is an edge so that v is not a cutvertex in G + e then v is not λ_i, κ_i, or μ_i-stable. But for the more sophisticated example in Figure 3.2, we see that $\lambda_i(v, G) = \kappa_i(v, G) = \mu_i(v, G) = 1$ and $\lambda_i(v, G + e) = \kappa_i(v, G + e) = \mu_i(v, G + e) = 2$ where e = xy. For an infinite class, this generalizes to an arbitrary cycle.

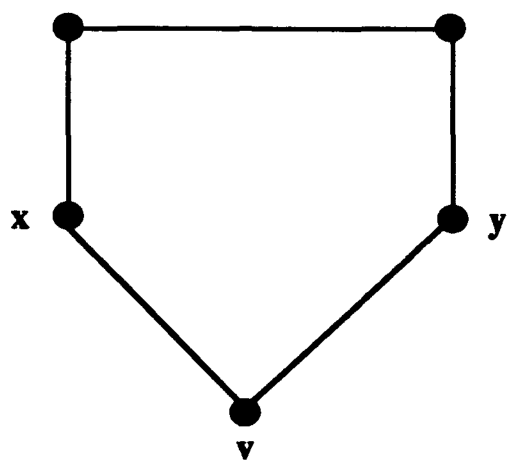


Figure 3.2 A vertex that is not λ_i, κ_i, or μ_i-stable.

Case (3): A vertex that is λ_i-stable, but not κ_i, or μ_i-stable.

For the graph in Figure 3.3, v is adjacent to every other vertex. By Theorems 2.10-2.12, none of the inclusive connectivity parameters for v can decrease under edge addition. It can be verified that $\lambda_i(v, G) = 3$ while $\kappa_i(v, G) = \mu_i(v, G) = 2$. The value of $\lambda_i(v, G)$ can be obtained in many ways but most notably by taking a neighborhood λ_i-set for v from any one of the three lightly shaded vertices. Since any edge added will be incident with at most two lightly shaded vertices, then one of the three neighborhood λ_i-sets for v in G will remain a valid neighborhood λ_i-set for v in G + e

for any edge $e \notin E(G)$. Thus v is λ_i-stable. We see that the addition of an edge between any of the nonadjacent lightly shaded vertices will cause the κ_i and μ_i values for v to increase to three. Therefore v is λ_i-stable, but not κ_i, or μ_i-stable.

One may notice that v is not the only vertex in the graph in Figure 3.3 that fits this case. The neighborhood of vertex u is complete, so $\lambda_i(u, G) = \kappa_i(u, G) = \mu_i(u, G) = 4$. Again, adding an edge between any of the nonadjacent lightly shaded vertices will give $\kappa_i(u, G + e) = \mu_i(u, G + e) = 3$ and $\lambda_i(u, G + e) = 4$. The stability of λ_i follows from the existence of at least four edge disjoint paths in G - u between any pair of neighbors of u after any edge addition, with several pairs of neighbors providing exactly four such paths. This will prevent λ_i from decreasing. An increase in λ_i will not occur, in a manner similar to vertex v.

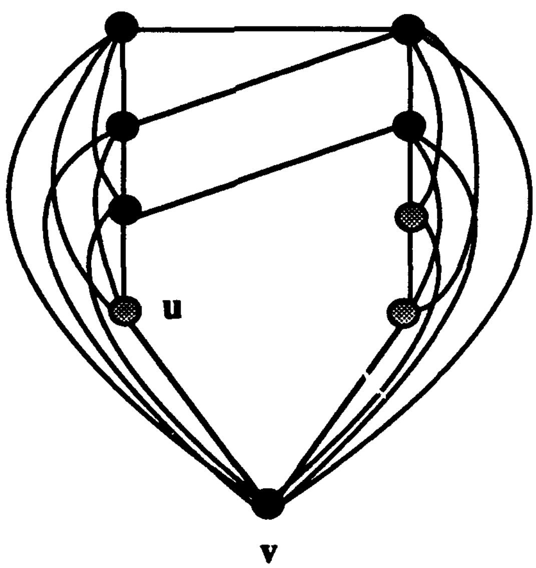


Figure 3.3 A vertex that is λ_i-stable, but not κ_i or μ_i-stable.

To construct an infinite class of graphs for this case, build graph G_n, $n \geq 4$ as follows. Let $G = (K_n \cup K_n) + v$, where + denotes the join operation. Label the

vertices of one copy of K_n $u_1, u_2, ..., u_n$ and the other $w_1, w_2, ..., w_n$. Define $V(G_n) = V(G)$ and $E(G_n) = E(G) \cup \{ u_1w_1 \} \cup \{ u_iw_{i-1}, i=2, 3, ..., n-1 \}$. Notice that the graphs G_n generalize the graph of Figure 3.3 and yield to a similar analysis.

Case (4): A vertex that is κ_i, and μ_i-stable, but not λ_i-stable.

If we take the join of a K_2 (naming the vertices w and v) with two additional copies of K_2 and two isolates (named u and x) we get the graph G in Figure 3.4. All three parameters for v have the value one. The only κ_i-set for v has the single member w. In G - v - w there are more than two components implying that for any $e \notin E(G)$, we will have a disconnected graph in (G + e) - v - w. Since (G + e) - w is connected, then $\kappa_i(v, G + e) \leq 1$. Noting that v cannot be a cutvertex in G + e, we have $\kappa_i(v, G + e) = \mu_i(v, G + e) = 1$. Thus v is κ_i- and μ_i-stable. But if we add the edge ux then the inclusive edge connectivity of v increases to two and we conclude that v is not λ_i-stable.

For an infinite class of such graphs, we take the join of a K_2 (w and v) with two additional K_n's, $n \geq 2$, and K_n - e, where e is an arbitrary edge, and the resultant graph behaves properly.

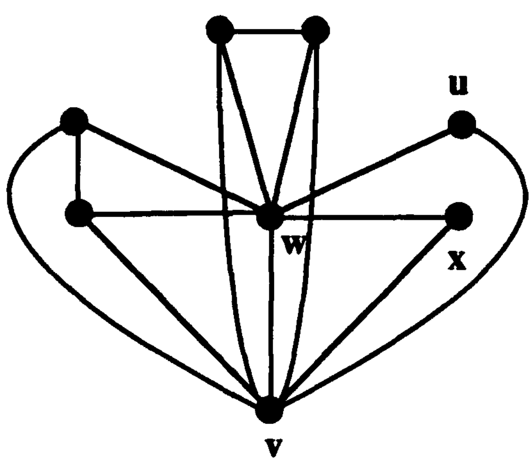


Figure 3.4 A vertex that is κ_i and μ_i-stable, but not λ_i-stable.

Case (5): A vertex that is μ_i-stable, but not λ_i or κ_i-stable.

Note that in the graph in Figure 3.5 ,G - v is constructed as $(K_3 - xy) + (K_3 - uw) - M$ where M is a certain matching between the two joined components. Since $\deg_{G-v}(u) = \deg_{G-v}(w) = 3$, we have $\mu_i(v, G) \leq 3$, $\kappa_i(v, G) \leq 3$, and $\lambda_i(v, G) \leq 3$. By counting the maximum number of internally disjoint paths in G - v between the vertices of $N_G(v)$, it can be shown that $\mu_i(v, G) = \kappa_i(v, G) = 3$, which also implies $\lambda_i(v, G) = 3$ since $\lambda_i(v, G) \geq \mu_i(v, G)$ and $\deg_{G-v}(u) = \deg_{G-v}(w) = 3$. Any edge $e \neq uw$ added to G - v will give $\mu_i(v, G + e) = \kappa_i(v, G + e) = 3$ since either vertex u or w will have degree 3 in G + e - v. If e = uw then $N_{G+e}(v)$ is now complete which implies $\kappa_i(v, G + e) = 4$. In addition, there are four edge disjoint paths between any pair of neighbors of v in G + uw - v so $\lambda_i(v, G + uw) = 4$. Thus v is not λ_i or κ_i-stable.

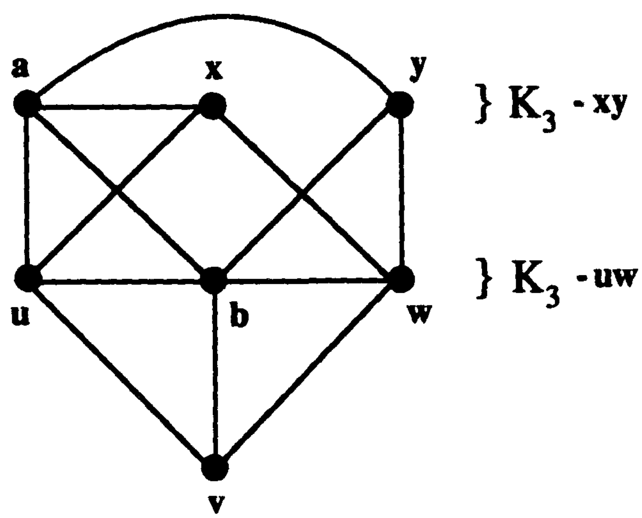


Figure 3.5 A vertex that is μ_i-stable, but not λ_i or κ_i-stable.

But $\mu_i(v, G + uw) \leq 3$ since {ub, a, w} is a μ_i-set for v, and combining this with Theorem 2.11 provides $\mu_i(v, G + uw) = 3$. To notice that the μ_i value does not

decrease in G + e when e is incident with v, we see that there are at least three internally disjoint paths from each of the vertices a, x, and y to each of the members of $N_G(v)$. Therefore $\mu_i(v, G + e) = 3$ for every $e \notin E(G)$ and v is μ_i-stable under edge addition.

Next we refer to the graph in Figure 3.6 to prove that there exists an infinite class of such graphs. Note that G - v in Figure 3.6 is constructed similarily to the graph in Figure 3.5 as $((K_n - x_{n-1}x_n) + (K_n - x_{n+1}x_{2n})) - M$ where the matching $M = \{ x_1x_{2n}, x_2x_{2n-1}, x_3x_{2n-2}, \cdots, x_ix_{2n-i+1}, \cdots, x_{n-1}x_{n+2}, x_nx_{n+1} \}$ and n is an integer, $n \geq 3$.

The degrees of x_{n+1} and x_{2n} in G - v are each (n - 2) + (n -1) = 2n - 3 while the degrees of $x_{n+2}, \ldots, x_{2n-1}$ in G - v are each (n - 1) + (n - 1) = 2n - 2. So by neighborhood sets, $\kappa_i(v,G) \leq 2n - 3$, $\mu_i(v, G) \leq 2n - 3$, and $\lambda_i(v,G) \leq 2n - 3$. To establish equality, we need to find at least 2n - 3 internally disjoint paths between every pair of vertices of $N_G(v)$ in G - v.

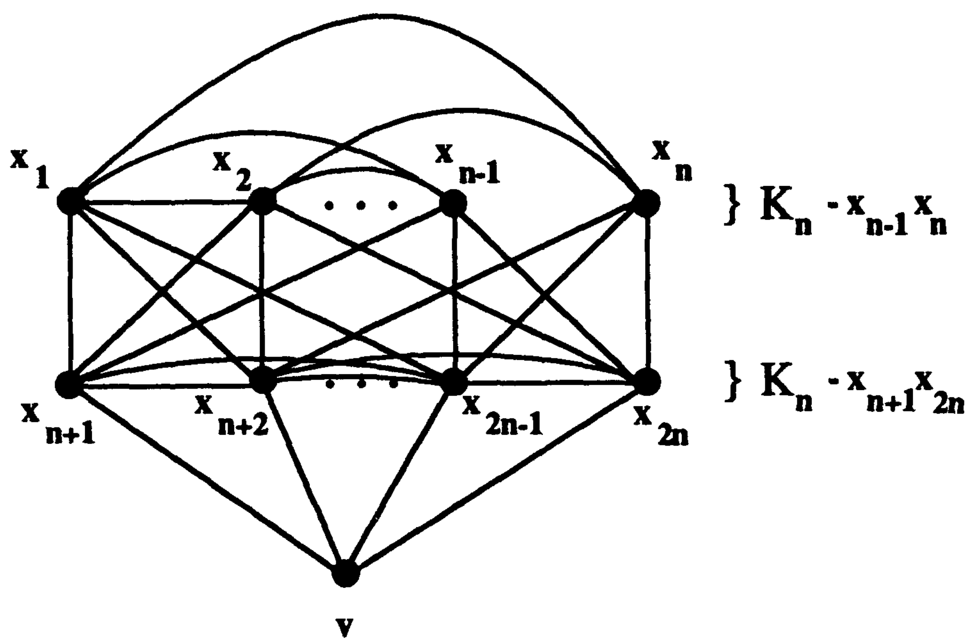


Figure 3.6 An infinite class for Case (5).

Let x_i and x_j be a pair of neighbors of v in G - v where $n + 1 \leq i < j \leq 2n$. Now we establish the required number of internally disjoint x_ix_j paths.

Case (5a): Let $i \neq n + 1$, $j \neq 2n$.

By inspection x_i is adjacent to all the vertices $x_{n+1}, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{2n}$ which are all adjacent to x_j. Thus we have n - 2 internally disjoint x_ix_j paths of length two and one more path of length one internally disjoint from the others.

Also, since $n + 1 < i < 2n$, x_i is adjacent to all the vertices $x_1, \cdots, x_{2n-i}, x_{2n-i+2}, \cdots, x_n$. A total of n - 2 of these vertices are adjacent to x_j for an additional n - 2 internally disjoint x_ix_j paths of length two. The vertex not adjacent to x_j is x_{2n-j+1}, which means the path $x_jx_{2n-j+1}x_{2n-i+1}x_j$ provides a total of 2n - 2 internally disjoint x_ix_j paths.

Case (5b): Let $i = n + 1$, $j = 2n$.

Now x_i is adjacent to $x_{n+2}, \cdots, x_{2n-1}$ which are all adjacent to x_{2n} for a total of n - 2 internally disjoint x_ix_j paths. But x_i is also adjacent to $x_1, x_2, \cdots, x_{n-1}$ which are all adjacent to x_j except x_1. But $x_ix_1x_nx_j$ provides another path for a grand total of (n - 2) + (n - 2) + 1 = 2n - 3 internally disjoint x_ix_j paths.

Case (5c): Let $i = n + 1$ or $j = 2n$ (but not both).

Without loss of generality, suppose $i = n + 1$ and $j \neq 2n$. Then x_i is adjacent to $x_{n+2}, \cdots, x_{2n-1}$ which are all adjacent to x_j (except x_j itself) for a total of n - 2 internally disjoint x_ix_j paths. But x_i is also adjacent to $x_1, x_2, \cdots, x_{n-1}$ which are all adjacent to x_j except vertex x_{2n-j+1}. However vertex x_{2n-j+1} is adjacent to x_{2n} which is adjacent to x_j for a path of length three. Thus we have a total of (n - 2) + (n - 2) + 1 = 2n - 3 internally disjoint x_ix_j paths.

Therefore, through the previous three cases, we have shown $\mu_i(v, G) \geq 2n - 3$, $\kappa_i(v, G) \geq 2n - 3$, and $\lambda_i(v, G) \geq 2n - 3$ which implies equality for all three parameters.

We must now complete our investigation by analyzing the change in the inclusive connectivity values for v after the addition of an arbitrary edge. We begin our investigation with the following claim:

Claim: The addition of any edge e except $x_{n+1}x_{2n}$ will keep $\mu_i(v, G + e) = \kappa_i(v, G + e) = \lambda_i(v, G + e) = 2n - 3$. But when $e = x_{n+1}x_{2n}$, then $\mu_i(v, G + e) = 2n - 3$ and $\kappa_i(v, G + e) = \lambda_i(v, G + e) = 2n - 2$.

The succeeding three cases substantiate this claim.

Case (5d): Add the edge $e = x_ix_j$ where $e \neq x_{n+1}x_{2n}$ and $x_i, x_j \neq v$ to the graph in Figure 3.6.

Since e is not incident with v we know the inclusive connectivity parameters for v remain the same or increase.

Since $e \neq x_{n+1}x_{2n}$ then one of x_{n+1} or x_{2n} will have degree 2n - 3 in G - v. Hence, $\mu_i(v, G + e)$, $\kappa_i(v, G + e)$, and $\lambda_i(v, G + e)$ remain 2n - 3.

Case (5e): Add the edge $e = x_{n+1}x_{2n}$ to the graph in Figure 3.6.

$N_{G+e}(v)$ is complete and the degree of each neighbor is 2n - 2, which implies $\kappa_i(v, G + e) = \lambda_i(v, G + e) = 2n - 2$. But $\mu_i(v, G + e) = 2n - 3$, since $S_m = \{x_{n+1}x_{n+2}, x_1, x_2, \cdots, x_{n-2}, x_{n+3}, x_{n+4}, \cdots, x_{2n}\}$ is a μ_i-set.

For clarity, $G - S_m$ is displayed in Figure 3.7.

Case (5f): Add the edge $e = x_iv$ where $1 \leq i \leq n$ to the graph in Figure 3.6.

By Theorems 2.10, 2.11, and 2.12, we know the inclusive connectivity parameters for v remain the same or decrease. Let $x_j \in N_G(v)$.

Let i = 1. So $n + 1 \leq j \leq 2n$ and x_1 is adjacent to $x_{n+1}, \cdots, x_{2n-1}$. And if $j \neq 2n$ then we have n - 1 internally disjoint x_1x_j paths of length at most 2, using the vertices $x_{n+1}, \cdots, x_{2n-1}$. Also x_1 is adjacent to $x_2, \cdots, x_n$ which provides n - 2 internally disjoint x_1x_j paths of length at most 2 for a total of 2n - 3 paths. If j = 2n, then we have n - 2 internally disjoint x_1x_{2n} paths of length at most 2 using vertices $x_{n+2}, \cdots, x_{2n-1}$ and n - 1 internally disjoint x_1x_{2n} paths of length at most 2 using vertices $x_2, \cdots, x_n$ for a total

of $2n - 3$. Thus if $i = 1$, then there exists at least $2n - 3$ internally disjoint paths between the new neighbor x_1 and all the members of $N_G(v)$, implying all the i-connectivity parameters remain the same.

Now let $2 \leq i \leq n - 2$. Then x_i is not adjacent to $x_{2n-i+1} \in N_G(v)$. If x_i is adjacent to x_j then there are $n - 1$ internally disjoint x_ix_j paths using the vertices $x_{n+1}, \cdots, x_{2n-i}, x_{2n-i+2}, \cdots, x_{2n}$. One of these paths may be of length three if $j = n + 1$ or $2n$ as follows. If $j = n + 1$, for example, then the path $x_ix_{2n}x_{2n-i+1}x_j$ suffices. Then the vertices $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$ provide $n - 2$ internally disjoint x_ix_j paths of length at most 2 for a total of $2n - 3$ paths. If x_i is not adjacent x_j, then we have $n - 1$ internally disjoint x_ix_j paths of length at most 2 utilizing the vertices $x_{n+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{2n}$. Now the vertices $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$ provide $n - 1$ internally disjoint x_ix_j paths of length at most 2 for a total of $2n -2$ paths. So for $2 \leq i \leq n - 2$, there exists at least $2n - 3$ internally disjoint paths between x_i and x_j.

Consider $i = n - 1$ or $i = n$. Without loss of generality let $i = n - 1$. Then we have $n - 1$ internally disjoint $x_{n-1}x_j$ paths of length at most 3 using $x_{n+1}, \cdots, x_{2n}$ where at most one path is of length three. Now x_{n-1} is adjacent to $x_1, x_2, \cdots, x_{n-2}$ which provides $n - 3$ internally disjoint $x_{n-1}x_j$ paths using the vertices $x_1, x_2, \cdots, x_{2n-j}, x_{2n-j+2}, \cdots, x_{n-2}$ for a total of $2n - 4$ paths. Finally we have the path $x_{n-1}x_{2n-j+1}x_nx_j$ for a total of $2n - 3$ internally disjoint x_ix_j paths where $i = n - 1$ and $n + 1 \leq j < 2n$. For the case $i = n$ the situation is similar.

Therefore, there exists at least $2n - 3$ internally disjoint paths between each $x_i \notin N_G(v)$ and $x_j \in N_G(v)$ which implies the inclusive connectivity parameters do not decrease but remain the same and the claim is established.

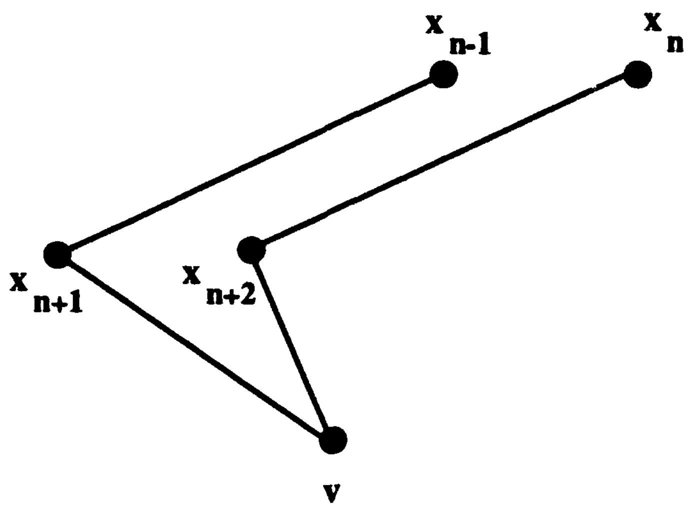


Figure 3.7 G - S_m for Case (5e).

Thus v ∈ V(G) in Figure 3.6 is μ_i-stable, but not κ_i or λ_i-stable and Case (5) is now complete. ❑

<u>Implications of i-Connectivity Stability under Edge Addition</u>

In this section we investigate implications that arise when considering the stability of inclusive connectivity for a given vertex. The relationships we consider are concerned with the following three major topics: global connectivity parameters, inclusive connectivity parameters, and stability under edge addition.

For example, it is known that $\kappa(G) > \kappa(G - v)$ if and only if v is in some minimum vertex separating set for G. First we present a result from [3].

Theorem 3.5: For any v ∈ V(G), if $\kappa(G) > \kappa(G - v)$ then $\kappa(G - v) = \mu_i(v) = \kappa_i(v) = \kappa(G) - 1$.

An example illustrating this theorem is presented in Figure 3.8. Here, $\kappa(G) = 2$ and $\kappa(G - v) = 1$ and by Theorem 3.5, $\mu_i(v) = \kappa_i(v) = 1$.

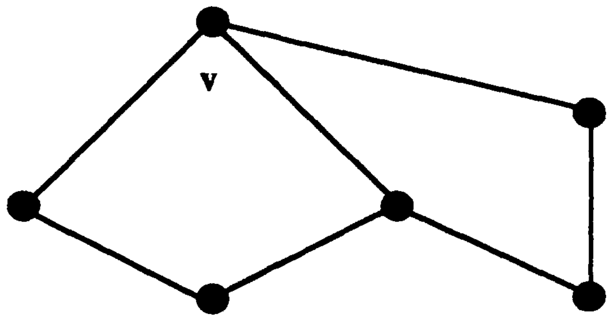


Figure 3.8 A graph illustrating Theorem 3.5.

The next corollary follows immediately.

Corollary 3.6: If $v \in V(G)$ is in some minimum vertex separating set for G, then $\kappa_i(v) = \mu_i(v)$.

The converse of Corollary 3.6 is not true as we can see from the graph in Figure 3.5 where $\kappa_i(v) = \mu_i(v) = 3$. But $\kappa(G) = 3 = \kappa(G - v)$, and hence, v is in no minimum vertex separating set for G.

The next two theorems describe when inclusive connectivities of a vertex can decrease.

Theorem 3.7: [22] Let G be a graph containing a vertex v such that $\lambda_i(v, G) > \lambda(G - v)$. Then there is a $w \in V(G)$, $w \notin N_G(v)$, so that $\lambda_i(v, G + vw) = \lambda(G - v)$.

Similar to Theorem 3.7 is the next theorem.

Theorem 3.8: [2] Let G be a graph containing a vertex v such that $\mu_i(v, G) > \kappa(G - v)$. Then there is a $w \in V(G)$, $w \notin N_G(v)$, so that $\mu_i(v, G + vw) = \kappa(G - v) = \kappa_i(v, G + vw)$.

These ideas can also be expanded to include κ_i, which we now do.

Theorem 3.9: Let G be a graph containing a vertex v such that $\kappa_i(v, G) > \kappa(G - v)$. Then there is a $w \in V(G)$, $w \notin N_G(v)$, so that $\kappa_i(v, G + vw) = \kappa(G - v)$.

Proof: Let S be a minimum vertex separating set of G - v. Since $\kappa_i(v, G) > \kappa(G - v)$, S does not separate two vertices from $N_G(v)$ into different components upon its

removal from G - v. Thus all of $N_G(v)$ resides in one component of G - v - S. Choose w to be a vertex from a component of G - v - S that does not contain the members of $N_G(v)$. Then v becomes a cutvertex in G + vw - S which means $\kappa_i(v, G + vw) \leq |S| = \kappa(G - v)$. Since $\kappa(G - v) \leq \kappa_i(v, G + vw)$ by Theorem 2.12(b), we have $\kappa_i(v, G + vw) = \kappa(G - v)$. ❑

Theorems 3.10-3.12 show that i-connectivity parameters produce global results for vertices that are stable.

Theorem 3.10: Suppose $v \in V(G)$ is λ_i-stable. Then $\lambda_i(v, G) = \lambda_i(v, G + e) = \lambda(G - v)$ for all $e \notin E(G)$.

Proof: Suppose $v \in V(G)$ is λ_i-stable. Suppose that $\lambda_i(v, G) > \lambda(G - v)$. Then by Theorem 3.7 there is a vertex w, $w \notin N_G(v)$ so that $\lambda_i(v, G + vw) = \lambda(G - v)$. But $\lambda_i(v, G + vw) = \lambda_i(v, G)$, a contradiction. Hence, $\lambda_i(v, G) \leq \lambda(G - v)$. But $\lambda_i(v, G) < \lambda(G - v)$ is impossible since any λ_i-set can be removed to disconnect G - v or leave it as the trivial graph. Therefore, the result follows. ❑

The proofs for Theorems 3.11 and 3.12 are similarly straightforward and have been omitted.

Theorem 3.11: Suppose $v \in V(G)$ is κ_i-stable under edge addition. Then $\kappa_i(v, G) = \kappa_i(v, G + e) = \kappa(G - v)$ for all $e \notin E(G)$.

Theorem 3.12: Suppose $v \in V(G)$ is μ_i-stable under edge addition. Then $\mu_i(v, G) = \mu_i(v, G + e) = \kappa(G - v)$ for all $e \notin E(G)$.

We have shown that an alternative procedure for obtaining inclusive connectivity values is to count the paths between neighbors of v. In the cases of increases in the λ_i and μ_i values for a vertex v when an edge $e \notin E(G)$ is added to a graph G, there is only one situation to consider. There must be respectively a new edge disjoint or internally disjoint path in G + e between a pair of neighbors previously separated in G - v - S where S is a λ_i-set or μ_i-set respectively.

In the case of κ_i, an increase could result from either the creation of a new path or by the edge joining a pair of neighbors previously separated now being adjacent in G + e. In any case, an increase in any i-connectivity parameter when an edge is added to a graph does not imply that either of its endpoints is a neighbor of the vertex. As can be seen in the graph in Figure 3.9, $\mu_i(v, G) = \kappa_i(v, G) = \lambda_i(v, G) = 1$ and $\mu_i(v, G + uw) = \kappa_i(v, G + uw) = \lambda_i(v, G + uw) = 2$, but $u, w \notin N_G(v)$.

Also, an increase in κ_i after the addition of edge e = uw implies that neither u nor w is in any κ_i-set for v. Otherwise, the addition of the edge would have no effect on the κ_i value, i.e., $\kappa_i(v, G + uw) = \kappa_i(v, G)$, since $G - v - S_v = (G + e) - v - S_v$.

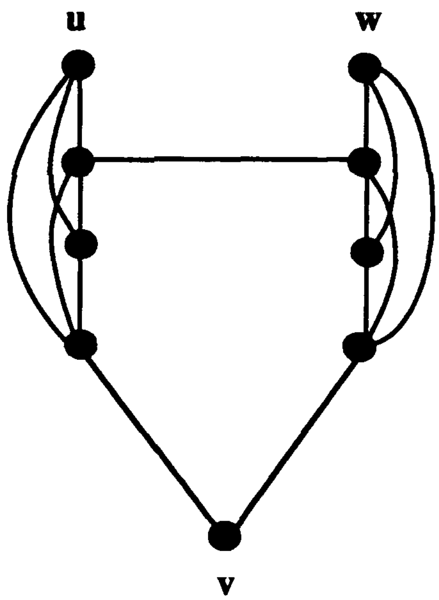


Figure 3.9 An increase in i-connectivity.

After initial work involving achievable relationships between i-connectivity parameters was accomplished in [4], Boland [2] asked whether one could find conditions under which the κ_i would "behave normally" (i.e., have $\kappa_i(v) \le \lambda_i(v)$,

paralleling Whitney's Theorem). It is interesting that the question can now be answered by using certain stability requirements. First we prove a lemma.

Lemma 3.13: If $v \in V(G)$ and $\deg(v) = |V(G)| - 1$, then $\mu_i(v) = \kappa(G - v) = \kappa_i(v) \leq \lambda_i(v) = \lambda(G - v)$.

Proof: Let $v \in V(G)$ and $\deg(v) = |V(G)| - 1$. If $\deg(v) = 1$, then $G = K_2$, and thus we have $\kappa_i(v) = \lambda_i(v) = \mu_i(v) = 0 = \kappa(G - v) = \lambda(G - v)$ if $\deg(v) = 0$ or 1. Suppose then that $\deg(v) \geq 2$. Since $\deg(v) = |V(G)| - 1$, then $N_G(v) = V(G - v)$ which implies $\lambda_i(v) = \lambda(G - v)$ and $\kappa_i(v) = \kappa(G - v) = \mu_i(v)$. Since by definition $\mu_i(v) \leq \lambda_i(v)$, the result follows. ❑

Lemma 3.13 implies that a vertex "behaves normally" if it is adjacent to every other vertex in the graph. We extend this to obtain a result when κ_i-stability is known.

Theorem 3.14: If $v \in V(G)$ is κ_i-stable under edge addition, then $\kappa_i(v, G) \leq \lambda_i(v, G)$.

Proof: Suppose $\kappa_i(v, G) > \lambda_i(v, G)$ and let S_e be a λ_i-set for v in G. We can assume $\deg(v) \geq 2$ since if $\deg(v) = 0$ then $\kappa_i(v, G) = \lambda_i(v, G) = 0$ and if $\deg(v) = 1$ then $\kappa_i(v, G) = \lambda_i(v, G) = \deg(u) - 1$ where $u \in N(v)$. Since $\deg(v) \geq 2$, $G - v - S_e$ has exactly two components and these components both contain vertices from $N(v)$. Let $u, w \in N(v)$ be in different components of $G - v - S_e$, say C_1 and C_2 respectively.

Case 1: Edge $uw \notin S_e$.

Add an edge e to G which is incident with v. Note that by the contrapositive of Lemma 3.13 such an edge exists. Since e is incident with v, $(G + e) - v - S_e$ has u and w in separate components just as $G - v - S_e$ did. Let S_v be a set of vertices which are endpoints of edges in S_e so that each edge has exactly one endpoint in S_v. Then in $(G + e) - v - S_v{*}$ u and w are neighbors of v which are in different components. Thus $\kappa_i(v, G + e) \leq |S_v{*}| \leq |S_e| = \lambda_i(v, G) < \kappa_i(v, G)$ which implies v is not κ_i-stable under edge addition.

Case 2: Edge $uw \in S_e$.

Claim: There exists a vertex in C_1 not adjacent to w, or a vertex in C_2 not adjacent to u. Assume to the contrary that w is adjacent to every vertex of C_1 and u is adjacent to every vertex of C_2. Then $\lambda_i(v, G) \geq | V(C_1) | + | V(C_2) | - 1 = | V(G) | - 2$. Thus G - $(V(C_1) \cup V(C_2) - u)$ isolates w and v in a K_2 component. Therefore, $\kappa_i(v, G) \leq | V(C_1) | + | V(C_2) | - 1 \leq \lambda_i(v, G)$, a contradiction which establishes the claim.

So assume there exists a vertex x in C_2 not adjacent to u. If x is not a neighbor of v in G, then we let e = vx. If x is a neighbor of v in G, then we let e be an edge (whose existence is assured by Lemma 3.13) from v to some other vertex of G not in the neighborhood of v. The same construction of S_v^* used in Case 1 can be used to separate the neighbors x and u of v in G + e. Then again $(G + e) - v - S_v^*$ separates neighbors u and x of v in G + e whereby $\kappa_i(v, G + e) \leq | S_v^* | \leq | S_e | = \lambda_i(v, G) < \kappa_i(v, G)$, which implies v is not κ_i-stable. ❑

Even though we now know that a vertex that is κ_i-stable does "behave normally", it can be easily demonstrated, that the converse is not true. In the graph in Figure 3.10, $\kappa_i(v, G) = 1$ and $\lambda_i(v, G) = 2$. But after the addition of edge e = uw we have $\kappa_i(v, G + e) = 4$ and $\lambda_i(v, G + e) = 3$, and thus v is not κ_i-stable.

We next explore a vertex possessing a unique λ_i, κ_i, or μ_i-set and being respectively λ_i, κ_i, or μ_i-stable. First we consider λ_i. In the case of K_3, every vertex has a unique λ_i-set, since for $v \in V(K_3)$, $\lambda_i(v) = 1$, and every vertex is trivially λ_i-stable under edge addition since we are dealing with a complete graph. Of course, we also have μ_i and κ_i-stability similarly.

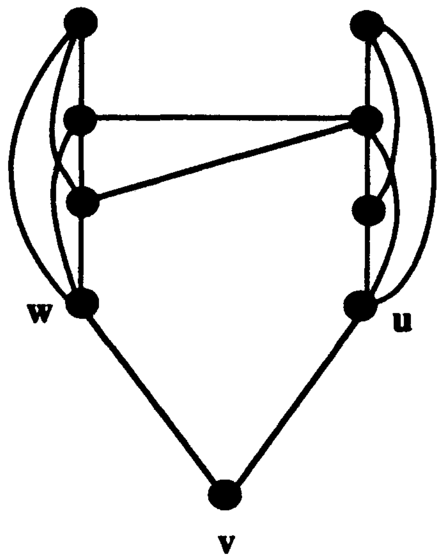


Figure 3.10 A vertex that "behaves normally" but is not κ_i-stable.

On the other hand, if $v \in V(G)$ where $\deg(v) = 1$, then v has a unique λ_i-set (the neighborhood λ_i-set from its lone neighbor) and it is possible for v to be λ_i-stable under edge addition. The graph in Figure 3.11 displays a vertex v of degree one that is λ_i-stable under edge addition. In this figure, $\lambda_i(v, G) = 4$ and $\lambda_i(v, G + e) = 4$ for any $e \notin E(G)$. Note that G - v is complete. We will prove later that this is a necessary condition for λ_i-stability under edge addition, for vertices of this degree.

One final special case is when a vertex is a cutvertex and will remain a cutvertex no matter what edge is added to the graph. As we see in the graph in Figure 3.12, $\lambda_i(v, G) = 0 = \lambda_i(v, G + e)$ for any $e \notin E(G)$, and since its unique λ_i-set is the null set, it also meets this criteria.

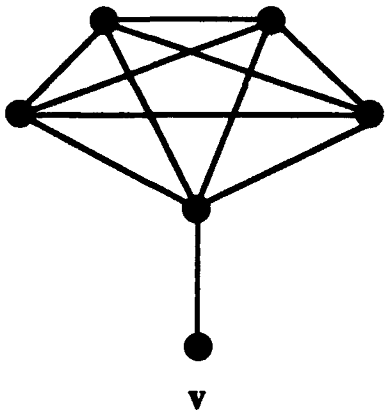


Figure 3.11 A vertex of degree one that is λ_i-stable.

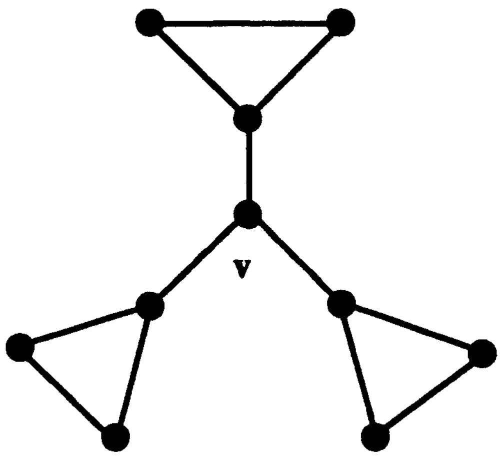


Figure 3.12 A cutvertex that is λ_i-stable.

Except for these three special cases, the next theorem shows that for $v \in V(G)$, if v has a unique λ_i-set, then it cannot be λ_i-stable.

Theorem 3.15: Let $G \neq K_3$ and $v \in V(G)$, $\deg(v) \geq 2$. If v has a nonempty unique λ_i-set, then v is not λ_i-stable.

Proof: Let $G \neq K_3$, and $v \in V(G)$, $\deg(v) \geq 2$, have a nonempty unique λ_i-set, S_e. Since $\deg(v) \geq 2$, then $G - v - S_e$ separates two neighbors u and w of v into exactly two components, C_1 and C_2, containing u and w respectively.

Case 1: In G there exists a vertex $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$ such that $v_1v_2 \notin V(G)$. Then add edge $e = v_1v_2$. Then $\lambda_i(v, G + e) = \lambda_i(v, G) + 1$, since $(G + e) - v - S_e$ is connected and S_e was a unique λ_i-set. Therefore, v is not λ_i-stable under edge addition.

Case 2: In G every vertex in C_1 is adjacent to every vertex in C_2. Then $\lambda_i(v, G) = | V(C_1) | * | V(C_2) |$ where * denotes standard multiplication. Since G is not K_3, either $| V(C_1) | > 1$ or $| V(C_2) | > 1$. Without loss of generality assume there exists a vertex $x \neq u$ in $V(C_1)$. Then $\deg(u) \leq | V(C_1) | - 1 + | V(C_2) | + 1$ counting edges to all vertices in $V(C_2)$, to all remaining $| V(C_1) | - 1$ vertices of $V(C_1)$ and the edge to vertex v. Thus, $\lambda_i(v, G) \leq \deg(u) - 1 \leq | V(C_1) | - 1 + | V(C_2) | + 1 - 1 \leq | V(C_1) | * | V(C_2) | = \lambda_i(v, G)$ since $| V(C_1) | \geq 2$, $| V(C_2) | \geq 1$ are positive integers. This implies that there exists an alternate λ_i-set, namely the neighborhood λ_i-set from vertex u, a contradiction. ❑

To show that a similar proposition for the remaining two i-connectivity parameters is not true, we direct the reader to the counterexample provided in Figure 3.13.

The graph in this figure is constructed as $G = (K_3 \cup K_3 \cup K_3) + K_2$ where + denotes the join operation and K_2 has the vertices v and w. The vertex v has a unique κ_i- and μ_i-set whose only element is the vertex w. But any possible edge addition must join a K_3 vertex to a vertex in another K_3. By Theorems 2.11 and 2.12, we know the values of κ_i and μ_i for v cannot decrease, so $\mu_i(v, G + e) = \kappa_i(v, G + e) = 1$, because $\{ w \}$ will remain a κ_i- and μ_i-set in $G + e$ for any $e \notin E(G)$. Thus it is possible to have a unique κ_i or μ_i-set and be κ_i or μ_i-stable under edge addition.

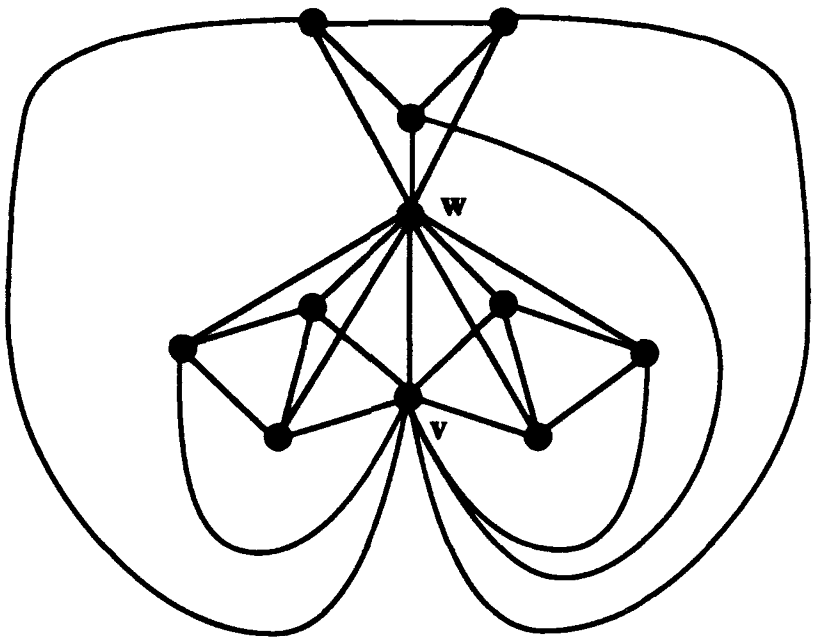


Figure 3.13 A vertex with a unique κ_i and μ_i-set.

Dependencies of Inclusive Connectivity under Edge Addition

The definition of stable inclusive connectivity simply states that the parameter does not change in value after any edge addition. No assumption whatsoever is implied about the sets of graph elements that produce these i-connectivity values. However, we can guarantee, for the addition of any edge e, the existence of a set of graph elements which is an i-connectivity set for v in both G and G + e.

Theorem 3.16: If $v \in V(G)$ is λ_i-stable, then for any $e \notin E(G)$ there exists a set of edges of G that is a λ_i-set for v in G and G + e.

Proof: Let $v \in V(G)$ be λ_i-stable and $e \notin E(G)$.

Case 1: Edge e is incident with v. Take any λ_i-set for v in G and call it S_e. Since $(G + e - v) = G - v$, we have $(G + e - v) - S_e = G - v - S_e$. This implies that the removal of S_e from $(G + e - v)$ will also separate the same two vertices from $N_G(v)$ into

different components. Since v is λ_i-stable under edge addition, then S_e is a λ_i-set for v in G + e.

Case 2: Edge e is not incident with v. Let S_e^* be a λ_i-set for v in G + e. Then (G + e) - v - S_e^* has vertices of $N_{G+e}(v)$ in different components. Since e is not adjacent to v, then $N_{G+e}(v) = N_G(v)$ and S_e^* will also separate the same two vertices from $N_{G+e}(v)$ into different components in G - v - S_e^*. Since v is λ_i-stable, S_e^* is a λ_i-set for v in G as well as G + e. ❑

The result for κ_i is now presented.

Theorem 3.17: If $v \in V(G)$ is κ_i-stable, then for any $e \notin E(G)$ there exists a set of vertices of G that is a κ_i-set for v in G and G + e.

Proof: Let $v \in V(G)$ be κ_i-stable and $e \notin E(G)$.

Case 1: Edge e is incident with v. Take any κ_i-set for v in G and call it S_v. So G - v - S_v either separates two vertices from $N_G(v)$ into different components or isolates a neighbor of v. Since (G + e - v) = G - v, we have (G + e - v) - S_v = G - v - S_v. This implies that the removal of S_v from (G + e - v) will also separate the same two vertices from $N_G(v)$ into different components or isolate the same neighbor of v. Since v is κ_i-stable, then S_v is a κ_i-set for v in G + e.

Case 2: Edge e is not incident with v. Take any κ_i-set for v in G + e and call it S_v^*. Then (G + e) - v - S_v^* either separates two vertices from $N_G(v)$ into different components or isolates a neighbor of v. Since e is not incident with v, then $N_{G+e}(v) = N_G(v)$ and S_v^* will also separate the same two vertices from $N_{G+e}(v)$ into different components or isolate the same neighbor of v. Since v is κ_i-stable, S_v^* is a κ_i-set for v in G as well as G + e. ❑

The proof for μ_i is similar and hence omitted.

Theorem 3.18: If $v \in V(G)$ is μ_i-stable, then for any $e \notin E(G)$ there exists a set of graph elements of G that is a μ_i-set for v in G and G + e.

We now turn our attention in beginning an investigation of edge addition and stability for κ_i. The κ_i parameter will provide us with some of the most surprising results among the inclusive connectivity parameters. This work began in [4] and so we will now present the most fundamental theorems utilized in this work. The next theorem, Boland's "One Edge Theorem", is arguably one of the most important for the study of the relationships between the i-connectivity parameters.

Theorem 3.19: For any $v \in V(G)$, if $\mu_i(v) < \kappa_i(v)$ then there exists a μ_i-set for v containing exactly one edge and that edge is between neighbors of v.

Note that this theorem makes no restriction on the values of the i-connectivities. It is quite surprising that even if the difference between the values of the two parameters are arbitrarily large, you are still guaranteed of the existence of a μ_i-set with exactly one edge!

Another useful result from [2] is Theorem 3.20.

Theorem 3.20: If $\mu_i(v) < \kappa_i(v)$ and S_m is any μ_i-set for v in G then $G - S_m - v$ has exactly two components which contain vertices of N(v).

When dealing with $\lambda_i(v)$, you are always certain to have exactly two components in $G - v - S_e$ for any λ_i-set, S_e, for v in G. You have no such guarantees when dealing with κ_i or μ_i since the deletion of a vertex from a graph can result in an arbitrarily large increase in the number of components. Theorem 3.20 is quite useful since it does provide, under certain conditions, these same guarantees in terms of the number of components containing neighbors of v.

In regards to Theorem 3.20, it is possible for a graph G to have more than two components in $G - S_m - v$. In the graph in Figure 3.14, $\mu_i(v) = 2$ with $S_m = \{ uw, x \}$ being a μ_i-set and $\kappa_i(v) = 3$ from a neighborhood κ_i-set from either u or w. But $G - v - \{ uw, x \}$ consists of three components where only two components contain vertices of $N_G(v)$ as guaranteed by this theorem.

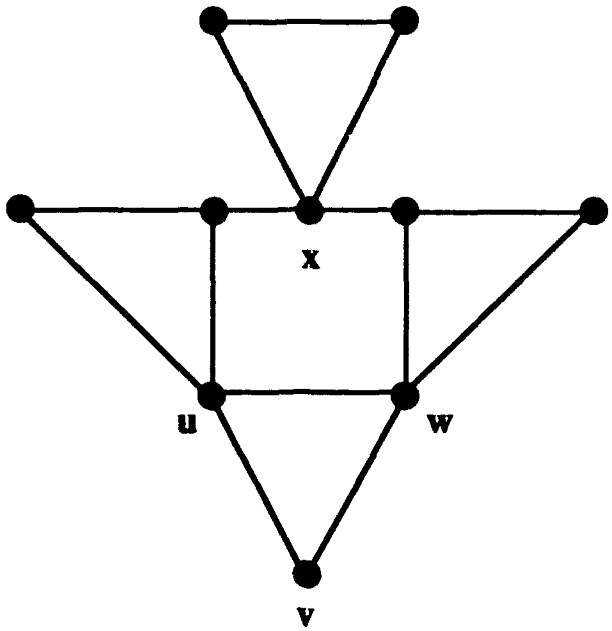


Figure 3.14 A graph illustrating Theorem 3.20.

We should note that the condition $\mu_i(v) < \kappa_i(v)$ does not imply that N(v) is complete. For example, in the graph in Figure 3.15, we have $\mu_i(v) = 3$ with the separation of neighbors y and z in G - v. And $\kappa_i(v) = 4$ with the separation of neighbors x and z in G - v, but N(v) is not complete.

We can establish that if $\mu_i(v) < \kappa_i(v)$ then there exists an edge e whose addition to G will cause κ_i to decrease to at most μ_i, which leads to an interesting relation to κ_i-stability. This situation is proven in Theorem 3.21.

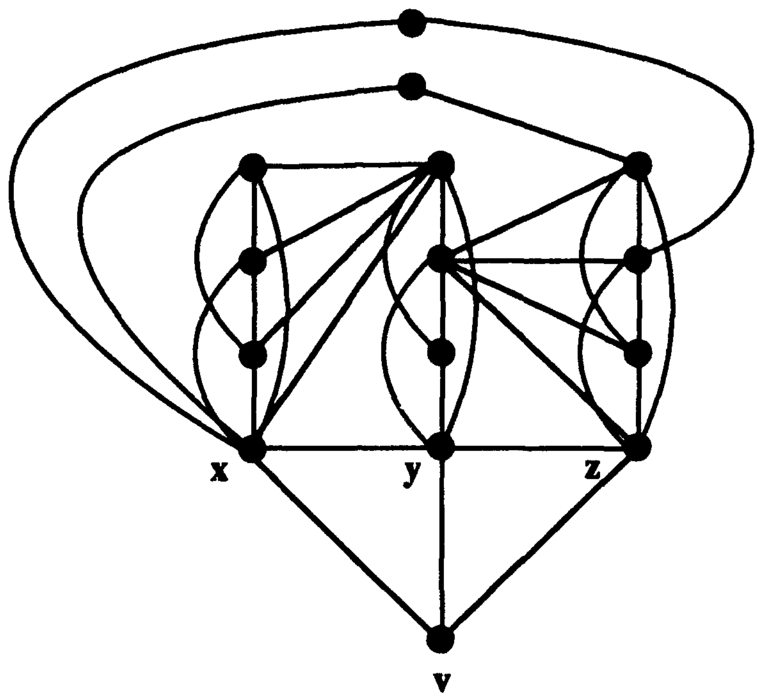


Figure 3.15 N(v) is not complete and $\mu_i(v) < \kappa_i(v)$.

Theorem 3.21: If $v \in V(G)$ satisfies $\mu_i(v, G) < \kappa_i(v, G)$ then v is not κ_i-stable under edge addition.

Proof: Let $v \in V(G)$ be such that $\mu_i(v, G) < \kappa_i(v, G)$. By Theorem 3.19 there exists a μ_i-set, S_m, for v in G with S_m containing exactly one edge $e = w_1w_2$ with w_1 and w_2 neighbors of v. Further, by Theorem 3.20, $G - S_m - v$ has exactly two components, C_1 and C_2, which contain vertices of N(v), and so assume $w_1 \in V(C_1)$ and $w_2 \in V(C_2)$. We may also assume, without loss of generality, that there is a vertex $x \in V(C_1)$ which is distinct from w_1. (If $|V(C_1)| = |V(C_2)| = 1$, then S_m is a neighborhood μ_i-set implying that $\mu_i(v, G) = \kappa_i(v, G)$.)

If $x \in N(v)$ then $S_m - e + w_1$ is a set of vertices whose removal from G makes v a cutvertex implying $\kappa_i(v, G) \leq |S_m - e + w_1| = |S_m| = \mu_i(v, G)$, a contradiction. Then it must be the case that $x \notin N(v)$.

Now consider the graph G + vx. The removal from G + vx of the set $S_m - e + w_1$ makes v a cutvertex. This gives $\kappa_i(v, G + vx) \le | S_m - e + w_1 | = | S_m | = \mu_i(v, G) < \kappa_i(v, G)$ so that v is not κ_i-stable in G. ❑

Corollary 3.22: If $v \in V(G)$ is κ_i-stable under edge addition in G then $\mu_i(v, G) = \kappa_i(v, G)$.

The construction technique used in the proof of Theorem 3.21 is illustrated in the graph in Figure 3.16. Here $\kappa_i(v, G) = 4$ while $\mu_i(v, G) = 2$ with one of the two μ_i-sets containing exactly one edge being $S_m = \{ w_1w_2, y \}$. If we add the edge vx to G, then we can remove edge w_1w_2 from S_m and replace it with the vertex w_1 thereby constructing a κ_i-set $S_v^* = \{ w_1, y \}$ for v in G + e which implies that the κ_i value has decreased by two from G to G + e.

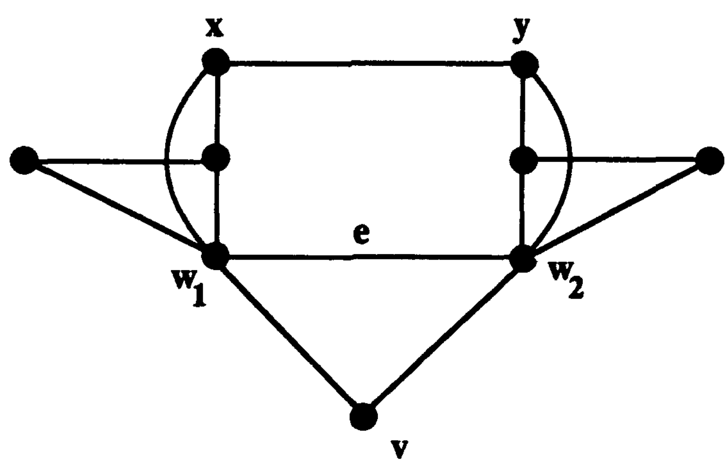


Figure 3.16 A graph illustrating Theorem 3.21.

This κ_i-stability result quickly leads us to the most important dependency of this chapter.

Theorem 3.23: If $v \in V(G)$ is κ_i-stable under edge addition in G then v is μ_i-stable under edge addition in G.

Proof: We establish the contrapositive, i.e., that if $v \in V(G)$ is not μ_i-stable under edge addition then v is not κ_i-stable under edge addition. Toward that end, suppose that $v \in V(G)$ is not μ_i-stable under edge addition. Let $e \notin E(G)$ with $\mu_i(v, G) \neq \mu_i(v, G + e)$.

Case 1: Suppose $\mu_i(v, G) = \kappa_i(v, G)$.

(a) If $\mu_i(v, G) < \mu_i(v, G + e)$ then $\kappa_i(v, G + e) \geq \mu_i(v, G + e) > \mu_i(v, G) = \kappa_i(v, G)$ implying that v is not κ_i-stable under edge addition.

(b) Suppose $\mu_i(v, G) > \mu_i(v, G + e)$. If $\kappa_i(v, G) \neq \kappa_i(v, G + e)$ then we are done. Assume then that $\kappa_i(v, G) = \kappa_i(v, G + e) > \mu_i(v, G + e)$. By Theorem 3.19, we let S_m be a μ_i-set for v in G + e such that S_m contains exactly one edge and that edge has as its endpoints neighbors of v. Since $\mu_i(v, G) > \mu_i(v, G + e)$, Theorem 2.11 implies that e must be adjacent to v. By Theorem 3.20, $(G + e) - S_m - v$ has exactly two components which contain neighbors of v. Let w_1 be a neighbor of v in component C_1 and w_2 be a neighbor of v in component C_2 of $(G + e) - S_m - v$. Let w_1w_2 be the edge in S_m. The proof proceeds in three subcases.

(i) If w_1 is the only vertex in C_1. Then $\mu_i(v, G + e) = \kappa_i(v, G + e)$. To verify this, notice that $S_m - w_1w_2 + w_2$ is a set of vertices whose removal from G + e makes v a cutvertex. Hence $\kappa_i(v, G + e) \leq | S_m - w_1w_2 + w_2 | = | S_m | = \mu_i(v, G + e)$. But $\kappa_i(v, G + e) \geq \mu_i(v, G + e)$ by definition, resulting in equality, a contradiction.

(ii) If there exists another vertex y in C_1 and $y \in N_{G+e}(v)$. Then v is a cutvertex with the removal of $S_m - w_1w_2 + w_1$ from G + e so $\kappa_i(v, G + e) = \mu_i(v, G + e)$, a contradiction as above.

(iii) If there exists another vertex y in C_1 and $y \notin N_{G+e}(v)$. Then consider the graph G + vy. Since $S_m - w_1w_2 + w_1$ is a set of vertices which makes v a cutvertex

upon removal from $G + vy$, $\kappa_i(v, G + vy) \leq | S_m - w_1w_2 + w_1 | = | S_m | = \mu_i(v, G + vy) < \mu_i(v, G) = \kappa_i(v, G)$, showing that v is not κ_i-stable under edge addition in G.

Case 2: Suppose $\mu_i(v, G) < \kappa_i(v, G)$. Then by Theorem 3.21, v is not κ_i-stable under edge addition in G. ❑

The reader is reminded that this is the result to which we alluded during the discussion of Theorem 3.4, regarding the relationships achievable for the stability of i-connectivity under edge addition. That is, it allowed us to reduce the number of possible combinations of stability among the parameters from eight to six. That theorem proved that μ_i-stability did not imply κ_i-stability. We will now investigate conditions under which this implication does hold true.

We begin with a complete analysis for vertices of degree one. So. let $v \in V(G)$ be such a vertex. In the graph in Figure 3.17 the neighborhood set around vertex u gives us $\lambda_i(v, G) = \kappa_i(v, G) = \mu_i(v, G) = 2$, and for $e = vy$ or vz we have $\lambda_i(v, G + e) = \kappa_i(v, G + e) = \mu_i(v, G + e) = 2$. Thus v is stable for all three i-connectivity parameters under edge addition.

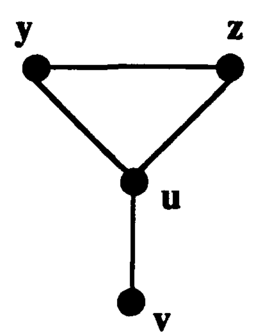


Figure 3.17 A degree one vertex that is λ_i, κ_i, and μ_i-stable.

On the opposite end of the spectrum, we can have a vertex of degree one that is none of λ_i, κ_i, or μ_i-stable. Vertex v in the graph in Figure 3.18 has $\lambda_i(v, G) = \kappa_i(v, G) = \mu_i(v, G) = 1$. But after adding the edge e = uz, $\lambda_i(v, G + e) = \kappa_i(v, G + e) = \mu_i(v, G + e) = 2$.

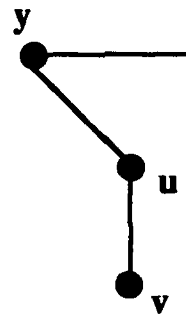


Figure 3.18 A degree one vertex that is not λ_i, κ_i, or μ_i-stable.

The next theorem will prove that these represent the only possible combinations of stability among the i-connectivity parameters for vertices of degree one.

Theorem 3.24: If $v \in V(G)$ has degree one, then v is λ_i, κ_i, and μ_i-stable or v is none of λ_i, κ_i, or μ_i-stable, depending on whether G - v is complete or not complete, respectively.

Proof: Let $v \in V(G)$ such that v has degree one. Assume that G - v is not complete and that the only neighbor of v is vertex u. If $\deg_{G-v}(u) \neq |V(G - v)| - 1$, then by adding an edge to G which is adjacent to u, we force each of $\lambda_i(v, G)$, $\kappa_i(v, G)$, and $\mu_i(v, G)$ to increase by 1. So unless $\deg_{G-v}(u) = |V(G - v)| - 1$, v is not λ_i-, κ_i-, or μ_i-stable under edge addition. If $\deg_{G-v}(u) = |V(G - v)| - 1$, then $\lambda_i(v, G) = \kappa_i(v, G) = \mu_i(v, G) = |V(G - v)| - 1$, which is the maximum for each parameter. In this case, if there exist two nonadjacent vertices in G - v, say x and y, neither one being u, then

the maximum number of internally disjoint paths between x and u in G - v is $|V(G - v)| - 2$, since $\deg_{G-v}(x) \leq |V(G - v)| - 2$. This implies that $\lambda_i(v, G + vx)$, $\kappa_i(v, G + vx)$, and $\mu_i(v, G + vx)$ will all decrease from $|V(G - v)| - 1$, and thus v is not λ_i, κ_i, or μ_i-stable.

Now assume that $G - v = K_n$ for some $n \geq 1$. In this case $\lambda_i(v, G) = \kappa_i(v, G) = \mu_i(v, G) = \deg_{G-v}(u) = |V(G - v)| - 1 = n - 1$. The only edges possible to add are incident with v. But, because G - v is a complete graph, the λ_i, κ_i, and μ_i values will remain n - 1, implying that v is λ_i, κ_i, and μ_i-stable. ❑

The desired result is now immediate.

Corollary 3.25: If $v \in V(G)$ is μ_i-stable under edge addition and $\deg_G(v) = 1$, then v is also κ_i-stable and λ_i-stable.

If we next examine Corollary 3.22 more carefully, we can easily show that its converse is not valid. In fact, the three following circumstances can occur:

(1) $\mu_i(v, G) = \kappa_i(v, G)$ with v both κ_i-stable and μ_i-stable under edge addition (any complete graph).

(2) $\mu_i(v, G) = \kappa_i(v, G)$ with v μ_i-stable, but not κ_i-stable under edge addition (see Figure 3.5).

(3) $\mu_i(v, G) = \kappa_i(v, G)$ and v is neither μ_i-stable nor κ_i-stable under edge addition (see Figure 3.16).

Therefore, where the μ_i and κ_i values for a vertex are the same, every possible case of stability between these two parameters can occur since Theorem 3.23 eliminates a possible fourth case.

But combining Theorem 3.14 and Corollary 3.22 we can narrow the number of possible relationships among the inclusive connectivity parameters to two.

Proposition 3.26: If $v \in V(G)$ is κ_i-stable under edge addition, then $\mu_i(v, G) = \kappa_i(v, G) \leq \lambda_i(v, G)$.

Both of these situations occur as evidenced in a complete graph where $\mu_i(v, G) = \kappa_i(v, G) = \lambda_i(v, G)$ and in the graph of Figure 3.4 where $\mu_i(v, G) = \kappa_i(v, G) < \lambda_i(v, G)$.

After the analysis of degree one vertices, it is tempting to suspect that whenever a vertex v of degree two is λ_i, κ_i, or μ_i-stable, then G - v must be a complete graph. That is, one might believe he could add an edge between the two components after the separation of the two neighbors of v, to "destroy" a λ_i, κ_i, or μ_i-set. If we examine the graph in Figure 3.19, we will see that this is not true. For this graph, an exhaustive analysis shows that $\lambda_i(v) = \kappa_i(v) = \mu_i(v) = 4 = deg_{G-v}(u) = deg_{G-v}(w)$. Since $deg_{G-v}(u) = deg_{G-v}(w) = 4$ there will be no increase in any of the parameters upon edge addition. However, every nonneighbor of v has exactly four internally (edge) disjoint paths to each neighbor of v, implying that adding an edge incident with v will create no change. Hence, v is λ_i, κ_i, and μ_i-stable, while G - v is not complete.

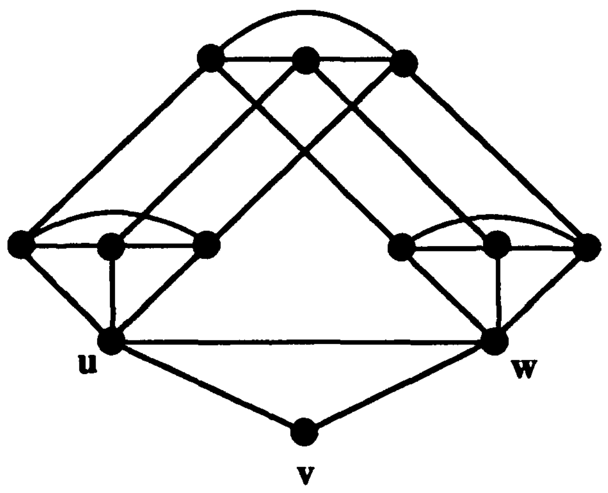


Figure 3.19 A vertex that is λ_i, κ_i, and μ_i-stable where G - v is not complete.

With some effort we further improve the previous condition for when μ_i-stability implies κ_i-stability. But first we must prove a lemma.

Lemma 3.27: If $v \in V(G)$ is μ_i-stable and every μ_i-set for v in G separates the same pair of neighbors of v, then $\mu_i(v, G) = \kappa_i(v, G)$.

Proof: Let $v \in V(G)$ be μ_i-stable and suppose every μ_i-set for v in G separates the same pair of neighbors, u and w. Since every μ_i-set separates u and w, and v is μ_i-stable, uw is in E(G) and is also in any μ_i-set for v in G.

Suppose that $\mu_i(v, G) < \kappa_i(v, G)$. Then no μ_i-set for v in G is a neighborhood μ_i-set for v at u or w. By Theorem 3.2, there exist exactly $\mu_i(v, G)$ internally disjoint u-w paths in G - v. Let S be any such set. Augment each path by using intermediate adjacencies, i.e., if $x_1x_2x_3 \cdots x_n$ is a path where $x_1 = u$, $x_n = w$, and x_i is adjacent to x_j where $i = 1, \cdots, n - 2$, $j = 3, \cdots, n$, $i < j - 1$, then adjust the path to become $x_1x_2 \cdots x_ix_j \cdots x_n$. Repeat this procedure until there are no such intermediate adjacencies. Thus, each path is "chordless".

Then there is at least one neighbor for each of u and w in G - v which is not on any path in S (otherwise v has a neighborhood μ_i-set at u or w). Call these neighbors x and y respectively. Notice that x and y are distinct since otherwise u-x-w is a u-w path not in S and internally disjoint from all paths in S, contradicting the fact that $\mu_i(v, G)$ is the maximum number of such u-w paths. Similarly, note that $xy \notin E(G)$. Then in G + xy, there are $\mu_i(v, G) + 1$ internally disjoint u-w paths implying that v is not μ_i-stable under edge addition. Then it must be the case that $\mu_i(v, G) = \kappa_i(v, G)$. ❑

Lemma 3.27 can be applied to the graph in Figure 3.20. Here $\mu_i(v, G) = 2$ and by inspection we see that v is μ_i-stable under edge addition since $\mu_i(v, G + e) = 2$ for any $e \notin E(G)$. And since there is only one pair of neighbors of v, we are guaranteed by this lemma that $\kappa_i(v, G) = 2$, which we can verify by inspection.

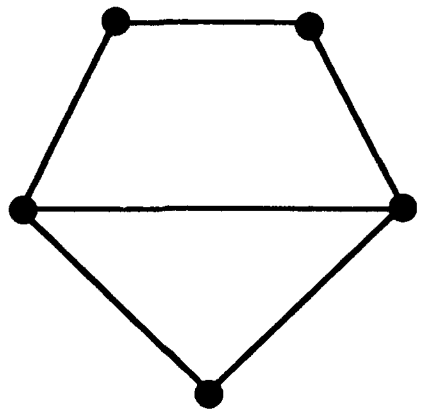


Figure 3.20 A graph illustrating Lemma 3.27.

Theorem 3.28: If $v \in V(G)$ is μ_i-stable and every μ_i-set for v in G separates the same pair of neighbors of v, then v is κ_i-stable.

Proof: Let $v \in V(G)$ be μ_i-stable and suppose every μ_i-set for v in G separates the same pair of neighbors of v, u and w. Then by Lemma 3.27, $\mu_i(v, G) = \kappa_i(v, G)$. Since v is μ_i-stable, then $uw \in E(G)$.

Case 1: Suppose $\deg_G(u) = \deg_G(w) = k$.

Claim: $\deg_{G-v}(u) = \deg_{G-v}(w) = \mu_i(v, G) = \kappa_i(v, G) = k$.

Assume for the sake of contradiction that $k \neq \mu_i(v, G) = \kappa_i(v, G)$. Then the number of internally disjoint u-w paths in G - v is strictly less than k. Let S be any set of $\mu_i(v, G)$ internally disjoint u-w paths each of which is "chordless" as discussed in Lemma 3.27. Then for each of u and w in G - v there is at least one neighbor which is not on any path in S. The technique used in Lemma 3.27 provides us with a new u-w path in G + e internally disjoint from the others for some $e \notin E(G)$, which contradicts the fact that v is μ_i-stable under edge addition, proving the claim.

Since $uw \in E(G)$, the degree of at least one of u or w remains the same when any edge e is added to G. It then follows that $\kappa_i(v, G + e) \leq \kappa_i(v, G)$. Combining this

with $\kappa_i(v, G + e) \geq \mu_i(v, G + e)$ and $\mu_i(v, G + e) = \mu_i(v, G)$ gives $\kappa_i(v, G + e) = \kappa_i(v, G)$ for all $e \notin E(G)$.

Case 2: With $\deg_G(u) \neq \deg_G(w)$ we assume without loss of generality $\deg_G(u) > \deg_G(w)$. Let P be any set of $\mu_i(v, G)$ internally disjoint u-w paths, each of which is chordless. Since $\mu_i(v, G) < \deg_G(u)$, there exists at least one neighbor x of u which is not on any path of P. Then $xw \notin E(G)$ since $\mu_i(v, G)$ represents the maximum number of internally disjoint u-w paths. Thus, in G + xw, there are $\mu_i(v, G) + 1$ internally disjoint u-w paths, contradicting the fact that v is μ_i-stable under edge addition. ❑

We now achieve an extension of Corollary 3.25 to the case of a degree two vertex.

Corollary 3.29: If $v \in V(G)$ is μ_i-stable and $\deg_G(v) = 2$, then v is κ_i-stable.

We note that any further extension of Corollaries 3.25 and 3.29 is not possible since Figure 3.5 provides us with an example of a vertex of degree three that is μ_i-stable but not κ_i-stable.

In contrast to Theorem 3.28, if every κ_i-set for v in G separates the same pair of neighbors, then v is guaranteed not to have κ_i-stability. This insures that all vertices of degree two where $u, w \in N(v)$, $uw \neq E(G)$, are not κ_i-stable.

Proposition 3.30: Given $v \in V(G)$, if every κ_i-set for v in G separates the same pair of neighbors, then v is not κ_i-stable.

Proof: Suppose every κ_i-set for v in G separates u and w into different components, where $u, w \in N_G(v)$. Since u and w are separated by a κ_i set then $uw \notin E(G)$. Thus $\kappa_i(v, G + uw) > \kappa_i(v, G)$ produces the desired result. ❑

The fact that a graph has a vertex v that has inclusive connectivity stability provides various details about the structure of the underlying graph G - v.

For example, if $v \in V(G)$ is μ_i-stable and $\deg(v) \geq 2$, then there must exist at least $\mu_i(v, G)$ internally disjoint paths in G - v between every pair of vertices x and y,

where $x \notin N(v)$, $y \in N(v)$. This is true because if otherwise, then we have $\mu_i(v, G + vx) < \mu_i(v, G)$ by Theorem 3.2.

Using the same reasoning we see that if $v \in V(G)$ is λ_i-stable and $deg(v) \geq 2$, then every vertex $x \notin N_G(v)$ must have at least $\lambda_i(v, G)$ edge disjoint paths to every neighbor of v in G - v .

We extend these ideas to include *any pair* of vertices in the graph, even if both vertices are not neighbors of the vertex v by using the concept of n-connectedness. A graph G is said to be *n-connected*, $n \geq 1$, if $\kappa(G) \geq n$. A graph G is *n-edge-connected*, $n \geq 1$, if $\lambda(G) \geq n$.

Theorem 3.31: If $v \in V(G)$ is μ_i-stable, then $G - v$ is $\mu_i(v, G)$-connected.

Proof: Let $v \in V(G)$ be μ_i-stable. Then by Theorem 3.12 we have $\mu_i(v, G) = \mu_i(v, G + e) = \kappa(G - v)$ for all $e \notin E(G)$. So by the definition of n-connected, $G - v$ is $\mu_i(v, G)$-connected. ❑

We now state Whitney's characterization of n-connected graphs [25].

Theorem 3.32: A nontrivial graph G is n-connected if and only if for each pair u, w of distinct vertices there are at least n internally disjoint u-w paths in G.

A corollary to Theorem 3.31 now follows by a direct application of Whitney's characterization of n-connected graphs, giving the desired extension.

Corollary 3.33: If $v \in V(G)$ is μ_i-stable, then there exists at least $\mu_i(v, G)$ internally disjoint paths between any pair of vertices of $G - v$.

A similar argument exists for the remaining two i-connectivity parameters with the λ_i parameter using edge disjoint paths.

Theorem 3.34: If $v \in V(G)$ is κ_i-stable, then $G - v$ is $\kappa_i(v, G)$-connected.

Corollary 3.35: If $v \in V(G)$ is κ_i-stable, there exists at least $\kappa_i(v, G)$ internally disjoint paths between any pair of vertices of $G - v$.

Theorem 3.36: If $v \in V(G)$ is λ_i-stable, then $G - v$ is $\lambda_i(v, G)$-edge-connected.

Corollary 3.37: If $v \in V(G)$ is λ_i-stable, there exists at least $\lambda_i(v, G)$ edge-disjoint paths between any pair of vertices of $G - v$.

Thus the stability of the inclusive connectivity parameters and their values for a specified vertex v provide information about connectivity in the graph $G - v$. For example one can examine a specific graph G and consider whether it is possible to add a vertex to G in such a way that the vertex has a type of inclusive connectivity stability. Thus revealing important connectivity information about the structure of G.

A counterexample is provided in Figure 3.21 to the converses of Theorems 3.31, 3.34, and 3.36. In this figure $\lambda_i(v, G) = \kappa_i(v, G) = \mu_i(v, G) = 2$ with $\kappa(G - v) = \lambda(G - v) = 2$. Also, $G - v$ is 2-connected ($\lambda_i(v, G)$-, $\kappa_i(v, G)$-, and $\mu_i(v, G)$-connected). But vertex v is not λ_i-, κ_i-, or μ_i-stable since if we add edge uw, we obtain $\kappa_i(v, G) = 4$, since $N_{G+uw}(v)$ is complete, and $\lambda_i(v, G) = \mu_i(v, G) = 3$.

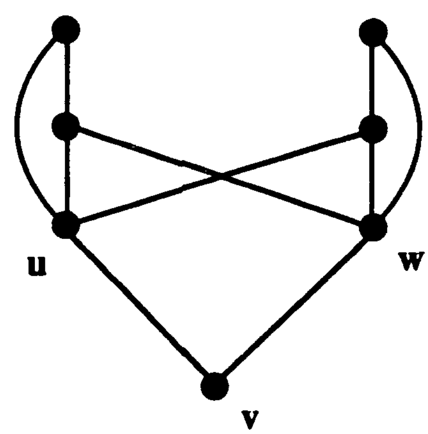


Figure 3.21 A counterexample to the converses of Theorems 3.31, 3.34, and 3.36.

The Relationship between the Stability of the Inclusive Connectivity Parameters and the Stability of the Global Parameters

After discussing the relationship between the stability of a given vertex v and the structure of the underlying graph $G - v$, we establish a surprising relation involving the

global connectivity parameters. We can show that inclusive connectivity stability implies the underlying G - v graph is also stable for the respective global parameter under edge addition!

Theorem 3.38: If $v \in V(G)$ is λ_i-stable, then $\lambda(G - v) = \lambda(G - v + e)$ for any $e \notin E(G - v)$.

Proof: Let $v \in V(G)$ be λ_i-stable. By Theorem 3.10 we know $\lambda_i(v, G) = \lambda_i(v, G + e) = \lambda(G - v)$ for any $e \notin E(G)$. Note that if e is adjacent to v in G + e, then the result holds since $G + e - v = G - v$. By viewing G + e as our graph, we have $\lambda_i(v, G + e) \geq \lambda((G + e) - v)$. Combining this with $\lambda((G + e) - v) \geq \lambda(G - v)$, we get $\lambda_i(v, G + e) = \lambda(G + e - v) = \lambda(G - v)$ which produces the desired result. ❑

By Theorem 3.38 we know that if $v \in V(G)$ is λ_i-stable under edge addition, then the global edge-connectivity for G - v is necessarily stable under edge addition also.

For the graph in Figure 3.22, it can be verified that $v \in V(G)$ is λ_i-stable under edge addition. So we know that the global edge connectivity of G - v, pictured in the graph of Figure 3.23, will not change when any edge is added.

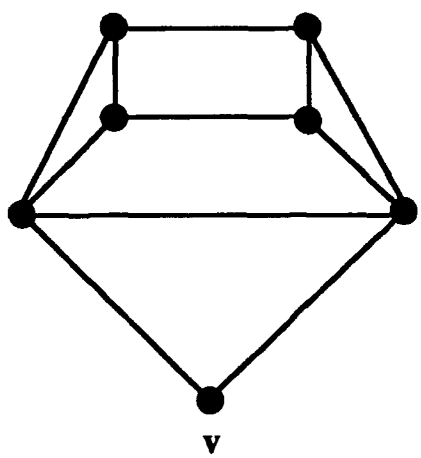


Figure 3.22 Relation of stability of i-connectivity and global parameters.

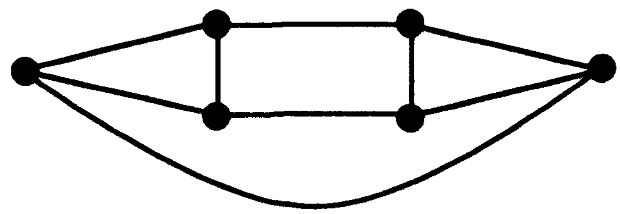


Figure 3.23 Edge connectivity remains unchanged upon edge addition.

We expand this notion of global stability to the other parameters as well. The proof of Theorem 3.39 is similar to Theorem 3.38 and is omitted.

Theorem 3.39: If $v \in V(G)$ is μ_i-stable under edge addition, then $\kappa(G - v) = \kappa(G - v + e)$ for any $e \notin E(G)$.

And finally, we have the result for κ_i which is a direct result of Theorem 3.23 and Theorem 3.39.

Theorem 3.40: If $v \in V(G)$ is κ_i-stable under edge addition, then $\kappa(G - v) = \kappa(G - v + e)$ for any $e \notin E(G)$.

These theorems instantly provide the power of identifying large classes of graphs whose edge and/or vertex stability does not change under edge addition. For example, for $G = K(n,n-1)$, $n \geq 3$, $\lambda(G) = \lambda(G + e)$ and $\kappa(G) = \kappa(G + e)$ for any $e \notin E(G)$. To see this, note that every vertex of $K(n,n)$ is λ_i-, and μ_i-stable under edge addition by Corollary 2.19.

Therefore, it is possible to obtain classes of graphs which are "maximal" with respect to edges and connectivity. This new type of graph can prove to be extremely interesting.

We now alternatively show that if a graph's edge or vertex connectivity does not change under edge addition, then it is a subgraph of a graph that contains a vertex that is λ_i- stable or κ_i and μ_i-stable under edge addition respectively.

Proposition 3.41: If G is such that $\lambda(G + e) = \lambda(G)$ for any $e \notin E(G)$, then $G + K_1$ has $v \in V(K_1)$, λ_i-stable, where + denotes the join operation.

Proof: Let G be a graph such that $\lambda(G + e) = \lambda(G)$ for any $e \notin E(G)$. Define $G^* = G + K_1$ where + is the join operation and $V(K_1) = \{ v \}$.

Let U be any edge disconnecting set for G. Then G - U has exactly two components. But $G^* - U = (G + v) - U$ is connected since v is adjacent to every vertex. So v is a cutvertex in $G^* - U$. Therefore $\lambda_i(v, G^*) \leq | U | = \lambda(G)$. Since $\lambda_i(u, G^*) \geq \lambda(G^* - u)$ for all $u \in V(G^*)$, then $\lambda_i(v, G^*) \geq \lambda(G)$, which implies $\lambda_i(v, G^*) = | U | = \lambda(G)$.

Now add any edge to G to get G + e and take any edge disconnecting set for G + e and call it U^*. Then $| U^* | = | U |$ since $\lambda(G + e) = \lambda(G)$ for all $e \notin E(G)$ and v is a cutvertex in $G^* + e - U^*$. Therefore $\lambda_i(v, G^* + e) \leq | U^* | = | U |$. But $| U | = \lambda(G) = \lambda_i(v, G^*)$, so $\lambda_i(v, G^* + e) \leq \lambda_i(v, G^*)$. But we know that e is not adjacent to v since v is not in V(G), so $\lambda_i(v, G^*) \leq \lambda_i(v, G^* + e)$ for all $e \notin E(G)$, implying equality. Therefore $\lambda_i(v, G^*) = \lambda_i(v, G^* + e)$ for all $e \notin E(G^*)$ implying v is λ_i-stable for edge addition in G^*. ❑

For the graph in Figure 3.24, $\lambda(G) = \lambda(G + e) = 2$ for any $e \notin E(G)$. Then by Proposition 3.41, $v \in V(K_1)$ is λ_i-stable in $G + K_1$.

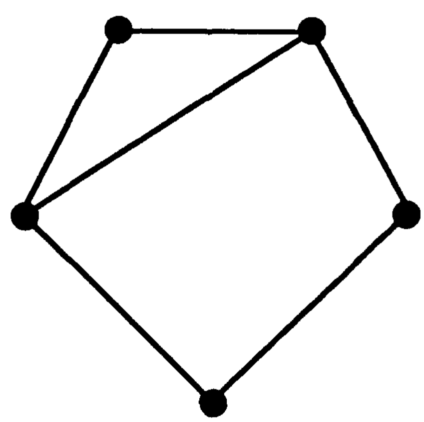


Figure 3.24 A graph where $\lambda(G + e) = \lambda(G)$.

The similar result for global vertex connectivity is now presented, and the proof follows the logic of the proof of Proposition 3.41 and is omitted.

Proposition 3.42: If G is such that $\kappa(G + e) = \kappa(G)$ for any $e \notin E(G)$, then $G + K_1$ has $v \in V(K_1)$, κ_i-stable, where + denotes the join operation.

A corollary to Proposition 3.42 provides the result for μ_i.

Corollary 3.43: If G is such that $\kappa(G + e) = \kappa(G)$ for any $e \notin E(G)$, then $G + K_1$ has $v \in V(K_1)$ μ_i-stable where + denotes the join operation.

Proof: By Proposition 3.42, v is κ_i-stable and thus by Theorem 3.23, v is μ_i-stable. ❑

CHAPTER 4

STABILITY OF INCLUSIVE CONNECTIVITY UNDER EDGE DELETION

Introduction

It is natural to now explore inclusive connectivity stability under edge deletion. We examine previous results concerning λ_i-stability under edge deletion for possible similar extensions to κ_i and μ_i. Ringeisen and Rice [20] studied this subject in relation to the results for λ_i [22].

Several different implications lead to interesting stability results under edge deletion. Throughout this chapter "stable" (or "stability") will mean "stable (stability) under edge deletion", i.e., that the relevant i-connectivity parameter does not change under edge deletion.

Extensions of Previous Results

A result similar to Theorem 2.10 concerning the behavior of λ_i after edge deletion was established by Rice [18].

Theorem 4.1: [18] Let v, u, and w be distinct vertices of G with deg(u) > 1 and deg(w) > 1. Let $e = uw \in E(G)$. Then

(a) $\lambda_i(v, G) - 1 \leq \lambda_i(v, G - e) \leq \lambda_i(v, G)$

(b) $\lambda(G - u) \leq \lambda_i(u, G) \leq \lambda_i(u, G - e)$.

This theorem implies that the change in the λ_i value after edge destruction is opposite of that under edge addition. Simply stated, if an edge e is deleted from a graph G and is not incident to $v \in V(G)$, then the λ_i value for v can only remain the same or decrease by exactly one. Because if e is contained in some λ_i-set, S_e, for v in G, then S_e - e will be a λ_i-set for v in G - e. On the other hand if e is incident to v, then

the λ_i value for v can only remain the same or increase. In this case $N_{G-e}(v) \subset N_G(v)$, which implies it is possible that a set of neighbors that used to get separated no longer are neighbors and hence some other pair must be separated causing λ_i to increase.

In the case where deg(u) = deg(w) = 1, the graph is K_2 and $\lambda_i(u, G) = \lambda_i(w, G) = 0$, with $\lambda_i(u, G - e) = \lambda_i(w, G - e) = 0$. If deg(u) = 1 and deg(w) > 1, then e = uw is a pendant edge. The graph G has $\lambda_i(u, G) = \deg(w) - 1$ and $\lambda_i(w, G) = 0$, but $\lambda_i(u, G - e) = 0$ and $\lambda_i(w, G - e)$ can have any value. Upon further examination, it is clear that these special cases also hold analogously for μ_i and κ_i.

The behavior of λ_i after edge deletion is extended to μ_i in Theorem 4.2.

Theorem 4.2: Let v, u, and w be distinct vertices of G with deg(u) > 1 and deg(w) > 1. Let $e = uw \in E(G)$. Then

(a) $\mu_i(v, G) - 1 \le \mu_i(v, G - e) \le \mu_i(v, G)$

(b) $\kappa(G - u) \le \mu_i(u, G) \le \mu_i(u, G - e)$.

Proof: Let S_m be a μ_i-set for v in G. Since e is not incident with v, then $N_G(v) = N_{G-e}(v)$. This implies that a pair of neighbors of v separated in $G - v - S_m$ will still be separated in $(G - e) - v - S_m$. Thus $\mu_i(v, G - e) \le | S_m | = \mu_i(v. G)$.

Now let S_m^* be a μ_i-set for v in G - e. Note that $e \notin S_m^*$. Then $S_m^* \cup \{ e \}$ is a set of graph elements that will separate some pair of neighbors of v in G as S_m^* did in G - e. This implies $\mu_i(v, G) \le | S_m^* | + 1 = \mu_i(v, G - e) + 1$ or $\mu_i(v, G) - 1 \le \mu_i(v, G - e)$ and (a) is proven.

For (b) note that G - e - u = G - u when e is incident with u so $\kappa(G - e - u) = \kappa(G - u)$ and also $N_{G-e}(u) \subset N_G(u)$. Let S_m^* be a μ_i-set for u in G - e. Then two neighbors of u are separated in $(G - e) - S_m^* - u$. Now two of the vertices of $N_G(u)$ in $G - S_m^* - u$ are separated, but since there are more vertices in $N_G(u)$ then in $N_{G-e}(u)$, then S_m^* may not be minimum. Hence $\mu_i(u, G) \le | S_m^* | = \mu_i(u, G - e)$. We have $\mu_i(u, G) \ge \kappa(G - u)$, thus $\kappa(G - u) \le \mu_i(u, G) \le \mu_i(u, G - e)$. ❑

The behavior of κ_i under edge deletion is analogous to the behavior of κ_i under edge addition and now follows.

Theorem 4.3: Let v, u, and w be distinct vertices of G with deg(u) > 1 and deg(w) > 1. Let $e = uw \in E(G)$. Then

(a) $\kappa_i(v, G - e) \leq \kappa_i(v, G)$

(b) $\kappa(G - u) \leq \kappa_i(u, G) \leq \kappa_i(u, G - e)$.

Proof: Let S_v be a κ_i-set for v in G. Since e is not incident with v, then $N_G(v) = N_{G-e}(v)$. This implies that a pair of neighbors of v separated or the neighbor isolated in $G - v - S_v$ will still be separated or isolated in $(G - e) - v - S_v$. Thus $\kappa_i(v, G - e) \leq |S_v| = \kappa_i(v. G)$. and (a) is proven.

To establish (b) note that $G - e - u = G - u$ since e is incident with u so $\kappa(G - e - u) = \kappa(G - u)$ and also $N_{G-e}(u) \subset N_G(u)$. Let S_v^* be a κ_i-set for u in G - e. Then two neighbors are separated or one neighbor of u is isolated in $(G - e) - (S_v^*) - u$. Now S_v^* will still separate or isolate the same neighbors of $N_{G-e}(u)$ upon removal from $G - u$, but since there are more elements in $N_G(u)$, then S_v^* may not be minimum. Hence $\kappa_i(u, G) \leq |S_v^*| = \kappa_i(u, G - e)$. Since we have $\kappa_i(u, G) \geq \kappa(G - u) = \kappa(G - e - u)$, we have $\kappa(G - u) \leq \kappa_i(u, G) \leq \kappa_i(u, G - e)$. ❑

As in edge addition, we notice in Theorems 4.1 and 4.2 that when λ_i and μ_i decrease, they can decrease by at most one. Yet Theorem 4.3 implies that this is not the case for κ_i and this situation of a decrease of more than one is illustrated in the graph of Figure 4.1.

In that figure, $\lambda_i(v, G) = \mu_i(v, G) = 1$ and $\kappa_i(v, G) = 6$. But $\lambda_i(v, G - e) = \mu_i(v, G - e) = \kappa_i(v, G - e) = 0$. It is clear from this example that it is possible for κ_i to decrease an arbitrary amount, even to its minimum value of 0, with the deletion of one edge. It is not a requirement that $< N(v) >$ be complete for this situation to occur. Figure 2.9 illustrates a case of $\kappa_i(v, G') = 8$ and $\kappa_i(v, G' - e) = 5$ where $G' = G + e$.

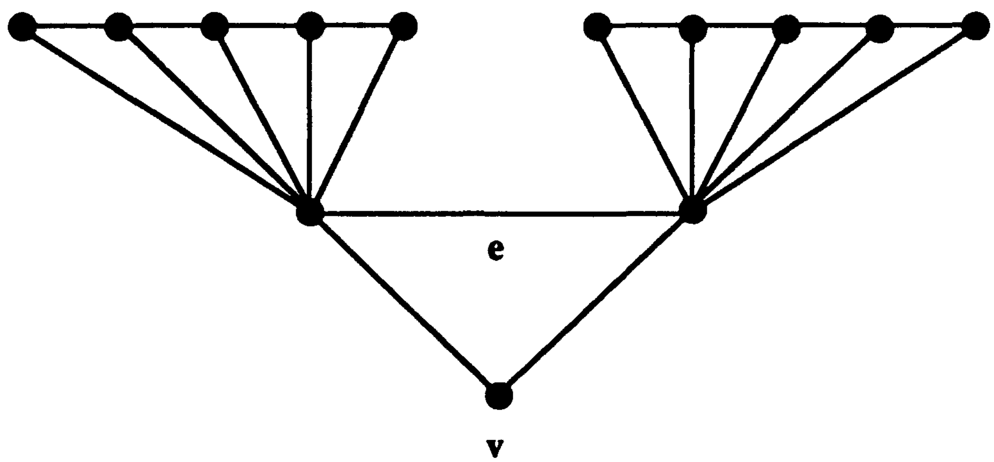


Figure 4.1 The decrease in κ_i after edge deletion.

An alternative viewpoint from that of Theorem 4.1(a) is that if an edge e is in a λ_i-set for a vertex, then in G - e the λ_i value will decrease by one; otherwise the λ_i value will remain the same. Rice formalized these conditions under which the λ_i value decreased.

Theorem 4.4: [18] Let v, u, and w be distinct vertices of G with $e = uw \in E(G)$. Then $\lambda_i(v, G - e) < \lambda_i(v, G)$ if and only if e is in some λ_i-set S_e for v in G.

Since Theorem 4.1 restricts the degree of the vertices of the deleted edge e, we consider those special cases for Theorem 4.4. This theorem will be vacously true for a cutvertex. For an endvertex, then we consider the case where $\deg(v) = 1$, $x \in N_G(v)$, and $e = uw$. Here, if x is not u or w then $\lambda_i(v, G) = \lambda_i(v, G - e) = \deg_G(x) - 1$. But if e is incident with x then w $\lambda_i(v, G) - 1 = \lambda_i(v, G - e) = \deg_G(x) - 2$.

We can show a similar situation for μ_i also exists.

Theorem 4.5: Let v, u, and w be distinct vertices of G with $e = uw \in E(G)$. Then $\mu_i(v, G - e) < \mu_i(v, G)$ if and only if e is in some μ_i-set S_m for v in G.

Proof: Let $v \in V(G)$. First, let e be in some μ_i-set S_m for v in G. Since $G - S_m = (G - e) - (S_m - e)$, then v is a cutvertex in $(G - e) - (S_m - e)$. Thus $\mu_i(v, G - e) \leq |S_m| - 1 = \mu_i(v, G) - 1$. Therefore, $\mu_i(v, G - e) < \mu_i(v, G)$.

Now let $\mu_i(v, G - e) < \mu_i(v, G)$. This implies $\mu_i(v, G - e) = \mu_i(v, G) - 1$. Let S_m^* be a μ_i-set for v in G - e. But v is a cutvertex of $(G - e) - S_m$ and $(G - e) - S_m = G - (S_m \cup \{ e \})$. Thus v is a cutvertex of $G - (S_m \cup \{ e \})$. So $|S_m| + 1 = \mu_i(v, G - e) + 1 = \mu_i(v, G)$ and S_m is a μ_i-set for v in G. ❑

Note that there is no restriction above in the degree of the vertex v since the case of an endvertex in Theorem 4.5 is identical to that in Theorem 4.4.

An analogous result for κ_i is not possible since an edge cannot be contained in a κ_i-set. It is natural to explore possibilities that may produce a characterization of when the κ_i value will decrease after edge deletion.

The mere presence of an edge e incident to a pair of neighbors of v being separated in G - e does not provide implications as to the behavior of κ_i.

In the graph of Figure 4.2 we have $\kappa_i(v, G) = 2$ and $\kappa_i(v, G - e) = 1$, and e is not incident to either of the neighbors of v that get separated in $G - e - S_v$, for a κ_i-set S_v. On the other hand, in the graph of Figure 4.3, edge e is incident to the two neighbors u and w of v being separated in $G - e - S_v$ where $S_v = \{ x, y \}$, but $\kappa_i(v, G) = \kappa_i(v, G - e) = 2$.

It now follows that the existence of an edge between neighbors of v is neither necessary nor sufficient to cause κ_i to change upon removal.

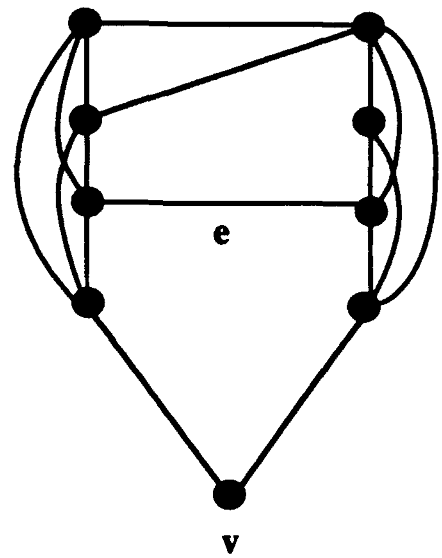


Figure 4.2 Edge e is not incident to the neighbors of v being separated.

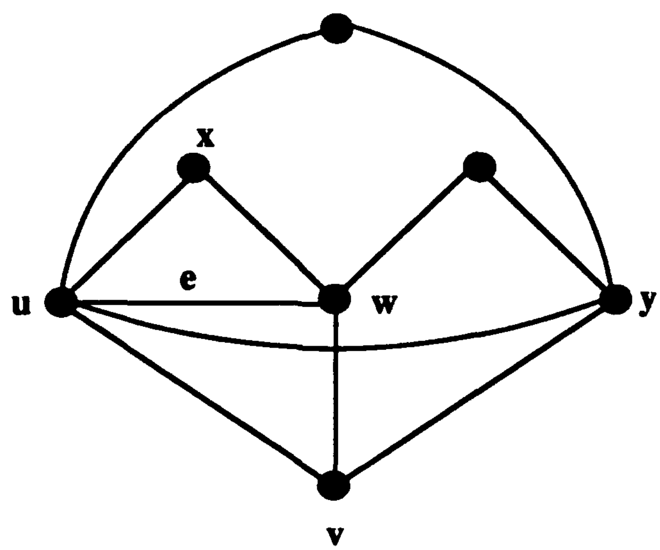


Figure 4.3 Edge e is incident to the neighbors of v being separated.

A more interesting possibility is the relationship between the decrease in κ_i value for a vertex v after the deletion of an edge e which is incident to a vertex that is in some κ_i-set for v. An edge e which is incident to a vertex of every κ_i-set is displayed in the graph of Figure 4.4. Here, the only κ_i-set for v in G is { x } and the κ_i value does not decrease since $\kappa_i(v, G) = 1 = \kappa_i(v, G - e)$.

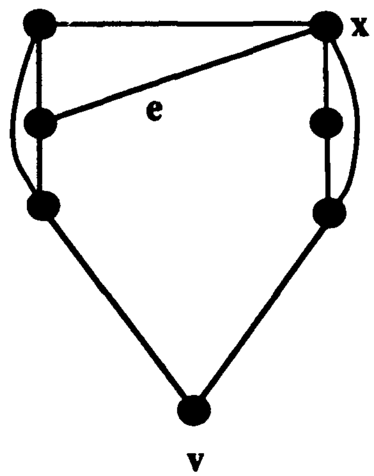


Figure 4.4 Edge e is incident to a vertex in every κ_i-set for v and $\kappa_i(v, G - e) = \kappa_i(v, G)$.

We can establish a partial relationship under a qualifying condition.

Theorem 4.6: For $v \in V(G)$ and $e \in E(G)$, if $\kappa_i(v, G - e) < \kappa_i(v, G)$ and the endpoints of e are not the only pair of neighbors separated in $G - e - S_v$ by every κ_i-set S_v for v in G - e, then e is incident to a vertex that is in some κ_i-set for v in G.

Proof: Let $\kappa_i(v, G - e) < \kappa_i(v, G)$ and let S_v be a κ_i-set for v in G - e that separates $u, w \in N_{G-e}(v)$ in $(G - e) - S_v - v$ where $e \neq uw$. Then $S_v^* = S_v \cup \{ x \}$, where x is an endpoint of e that is not u or w, will separate u and w in $G - S_v^* - v$. So $\kappa_i(v, G) \leq \kappa_i(v, G - e) + 1$ and since $\kappa_i(v, G - e) < \kappa_i(v, G)$ implies $\kappa_i(v, G) \geq \kappa_i(v, G - e) + 1$,

we have $\kappa_i(v, G) = \kappa_i(v, G - e) + 1$. Therefore S_v* is a κ_i-set and e is incident with a vertex that is in some κ_i-set for v in G. ❑

The qualifying condition in Theorem 4.6 eliminates the case where the deletion of an edge from G results in a pair of neighbors previously adjacent in G providing the sole determination (and decrease) of the κ_i value in G - e. The graph in Figure 4.5 illustrates that this condition is required for the result in the previous theorem. In this figure, an edge e is removed from G and the κ_i value for v drops, where there is only one pair of neighbors being separated; namely the endpoints of e, and neither endpoint of e is in any κ_i-set for v in G.

First we establish that $\kappa_i(v, G - e) < \kappa_i(v, G)$. All the i-connectivity values here were confirmed by using the algorithm from [12]. A κ_i-set for v in G consists of the six unlabeled vertices incident to vertex x (or y) while a κ_i-set for v in G - e consists of the three lightly shaded vertices and the vertices x and y. Thus the κ_i value for v decreases upon the removal of e from G.

To prove that u or w is not in any κ_i-set we argue the following: Examine G - u (or G - w) for the number of internally disjoint paths between the neighbors of v, which in effect is putting u (or w) in a κ_i-set. In G - u (or G - w) there exists six internally disjoint x-y paths, u-x (w-x) paths, and u-y (w-y) paths. Therefore, u or w cannot be in any κ_i-set for v in G.

A characterization of when the κ_i value decreases for a vertex is possible if we consider vertex deletion. This condition is given in Theorem 4.7.

Theorem 4.7: For v, u $\in$ V(G), $\kappa_i(v, G - u) < \kappa_i(v, G)$ if and only if u is in some κ_i-set for v in G.

Proof: Let $\kappa_i(v, G - u) < \kappa_i(v, G)$ which implies $\kappa_i(v, G - u) \leq \kappa_i(v, G) - 1$. Let S_v* be a κ_i-set for v in G - u. If (G - u) - S_v* - v separates neighbors x and y of v, then for G, G - (S_v* $\cup$ { u }) - v will also separate the neighbors x and y, since G - u - S_v* =

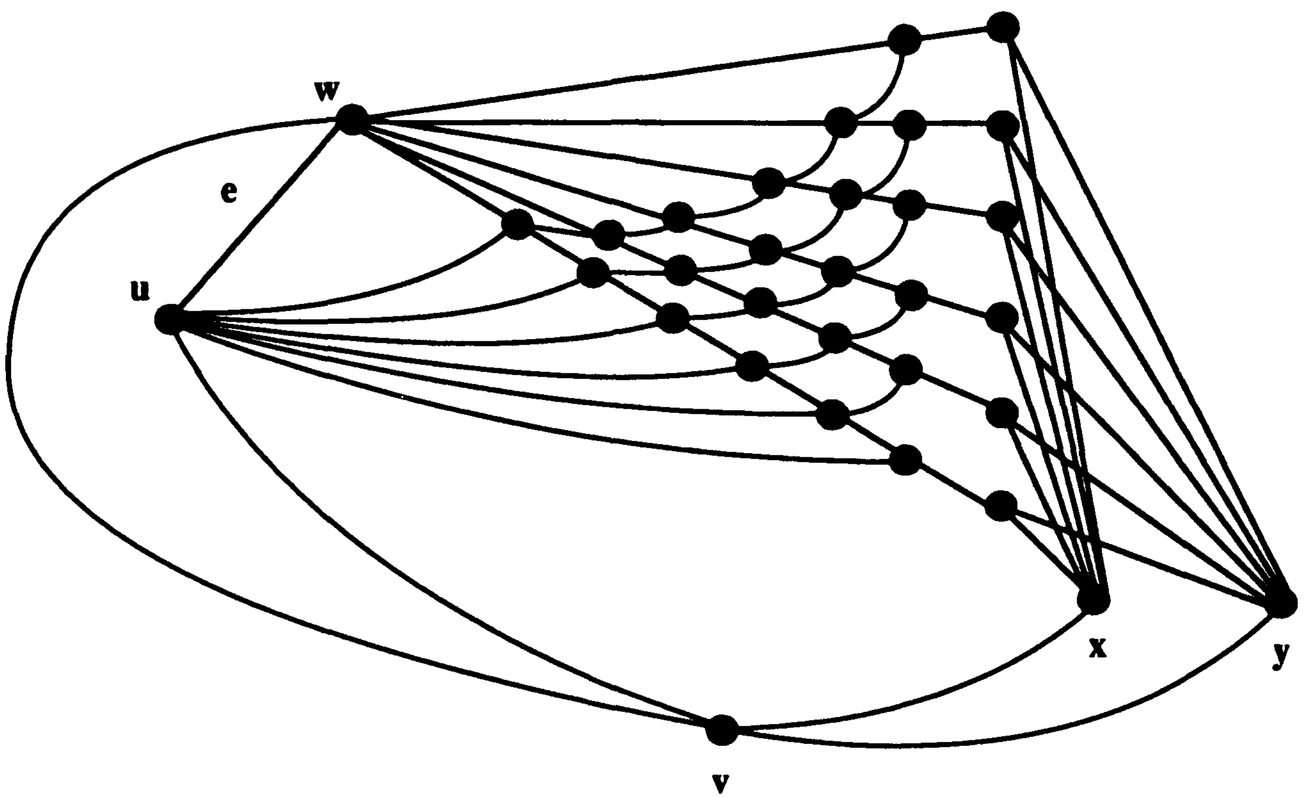


Figure 4.5 A vertex v that has only one pair of neighbors separated in G - e.

$G - (S_v^* \cup \{u\})$. But $|S_v^* \cup \{u\}| = \kappa_i(v, G - u) + 1$ which means $S_v^* \cup \{u\}$ is a κ_i-set for v in G. Thus, u is in some κ_i-set for v in G. If $(G - u) - S_v - v$ isolates a neighbor of v, then $S_v^* \cup \{u\}$ will do likewise.

Let $u \in V(G)$ be in some κ_i-set S_v for v in G. Then $G - v - S_v$ either separates neighbors x and y of v or isolates neighbor z of v. In either case, $S_v - u$ will still separate neighbors x and y of v or isolate neighbor z of v in G - u since $(G - u) - v - (S_v - u) = G - v - S_v$. Therefore $\kappa_i(v, G - u) \leq |S_v - u| = \kappa_i(v, G) - 1$ which implies $\kappa_i(v, G - u) < \kappa_i(v, G)$. ❑

A characterization of when the deletion of an edge can increase the λ_i value of a vertex is presented next. For a proof, see [18].

Theorem 4.8: Let G be a connected graph and let e = uv with $\lambda_i(v, G) \neq 0$, $\lambda_i(u, G) \neq 0$, and $\deg(v) \geq 3$. For $v \in V(G)$, $\lambda_i(v, G - e) > \lambda_i(v, G)$ if and only if e = uv is a bridge in $G - S_e$ for every λ_i-set S_e for v in G.

The previous result does not hold for those special cases excluded in the hypothesis of the theorem [18]. For instance, the λ_i value for a cutvertex increases if and only if after removal of the edge e it is no longer a cutvertex. If e is a pendant edge, then the λ_i value of the endvertex decreases from some positive λ_i value to zero. And for $\deg(v) = \deg(u) = 2$ in C_n, then edge $e = uv \in E(C_n)$ is a bridge in $G - S_e$ for every λ_i-set for u and v, but the λ_i value for neither increases when e is deleted.

A variation of Theorem 4.8 for κ_i is given in Theorem 4.9.

Theorem 4.9: Let G be a connected graph and let e = uv with $\kappa_i(v, G) \neq 0$, $\kappa_i(u, G) \neq 0$, and $\deg(v) \geq 3$. For $v \in V(G)$, if $\kappa_i(v, G - e) > \kappa_i(v, G)$ then e = uv is a bridge in $G - S_v$ for every κ_i-set S_v for v in G.

Proof: Let G be a connected graph where e = uv and $\kappa_i(v, G) \neq 0$, $\kappa_i(u, G) \neq 0$, and $\deg(v) \geq 3$. Suppose $\kappa_i(v, G - e) > \kappa_i(v, G)$ and e is not a bridge in $G - S_v$ for some κ_i-set S_v of v in G.

First assume that $G - v - S_v$ isolates a neighbor of v. Then that neighbor can only be u since $\kappa_i(v, G - e) > \kappa_i(v, G)$. But this makes e a bridge in $G - S_v$, a contradiction.

So assume $G - v - S_v$ has at least two components, each of which contains at least one neighbor of v. Note that $u \notin S_v$, because if $u \in S_v$ then $G - v - S_v = (G - e) - v - S_v$ and S_v will separate the same neighbors in $G - v - S_v$ as in $(G - e) - v - S_v$ which implies $\kappa_i(v, G - e) \le \kappa_i(v, G)$, a contradiction.

Since $u \notin S_v$, then u is in one of the components of $G - v - S_v$ Let C_1 and C_2 be two components of $G - v - S_v$ where $u \in C_1$ and $x \in C_2$, $x \in N_G(v)$. There must exist a vertex $z \in N_G(v)$ where $z \neq u$ so that $z \in C_1$. Otherwise uv is a bridge in $G - S_v$. Since $(G - e) - v - S_v = G - v - S_v$, $G - e$ is divided into the same components in $(G - e) - v - S_v$ with $z \in C_1$, $x \in C_2$, and $z, x \in N_{G-e}(v)$. Therefore S_v separates the neighbors of v in $G - e - v$ into different components which implies $\kappa_i(v, G - e) \le \kappa_i(v, G)$, a contradiction. Thus e is a bridge in $G - S_v$ for every κ_i-set S_v of v in G. ❑

In contrast to Theorem 4.8, the converse of Theorem 4.9 is not true. Vertex v in the example shown in Figure 4.6 has only one κ_i-set, namely $S_v = \{ x \}$.

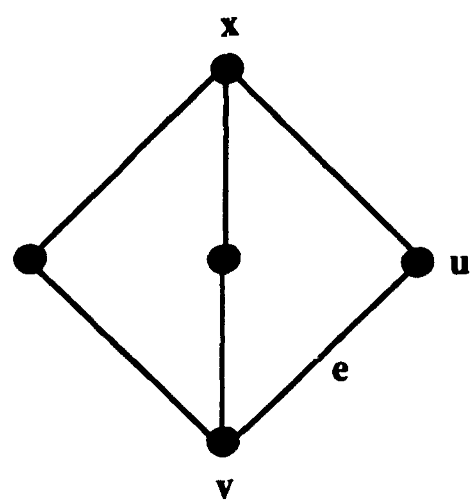


Figure 4.6 Edge e is a bridge in $G - S_v$ for every κ_i-set S_v for v in G.

Here, in G - S_v the edge e is clearly a bridge while in G - e we have $\kappa_i(v, G - e) = 1$. Therefore, it is possible for an edge e to be a bridge in G - S_v for every κ_i-set S_v for v in G and $\kappa_i(v, G - e) = \kappa_i(v, G)$.

Except for the situation of isolating a neighbor of v, the proof of Theorem 4.10 for μ_i is similar to Theorem 4.9 and is omitted.

Theorem 4.10: Let G be a connected graph and let e = uv with $\mu_i(v, G) \neq 0$, $\mu_i(u, G) \neq 0$, and $\deg(v) \geq 3$. For $v \in V(G)$, if $\mu_i(v, G - e) > \mu_i(v, G)$ then e = uv is a bridge in G - S_m for every μ_i-set S_m for v in G.

Again the converse of the previous theorem does not hold. In the graph of Figure 4.7, $\mu_i(v, G) = 1$ with only one μ_i-set, $S_m = \{ x \}$. Here, e is a bridge in G - S_m for every μ_i-set S_m, but $\mu_i(v, G - e) = 1$.

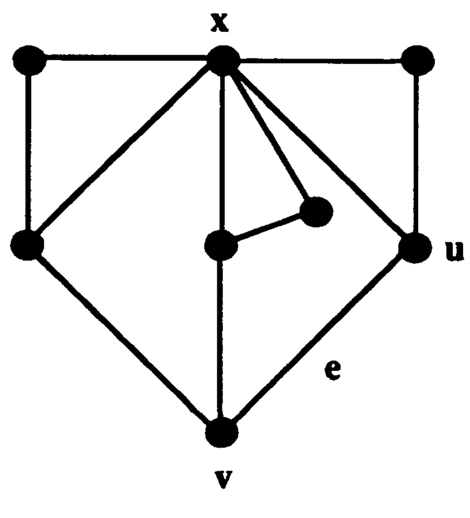


Figure 4.7 A counterexample to the converse of Theorem 4.10.

Implications of i-Connectivity Stability under Edge Deletion

Before investigating i-connectivity stability under edge deletion, we first make some elementary observations. If an edge is contained in a λ_i-set for a vertex v, then

the edge is not incident to v. Thus, a vertex v with $\lambda_i(v) > 0$ cannot be λ_i-stable under edge deletion since the removal of any edge from a λ_i-set for v will decrease the λ_i value by one. Therefore, we concentrate on such stability properties for μ_i and κ_i only.

Likewise, a vertex v with $\mu_i(v,G) > 0$ cannot be μ_i-stable under edge deletion if any μ_i-set for v contains an edge. If every μ_i-set for v has only vertices, then it is possible for v to be μ_i-stable, as illustrated in the graph of Figure 4.8.

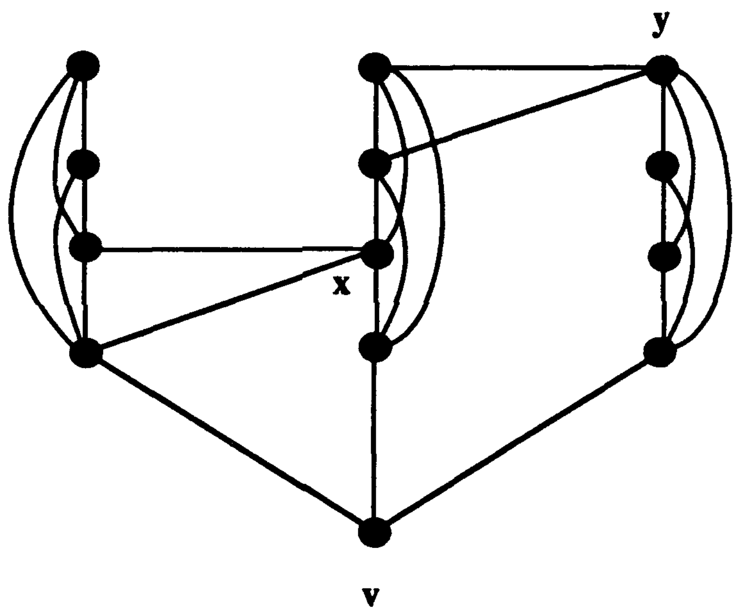


Figure 4.8 A vertex that is μ_i and κ_i-stable under edge deletion.

For the graph in this figure, $\kappa_i(v, G) = \mu_i(v, G) = 1$ with the two possible κ_i and μ_i-sets being { x } and { y }. There are two edge disjoint paths between every pair of neighbors of v in G - v. This implies that deleting an edge will leave at least one path between every pair of neighbors of v in G - v. Thus $\kappa_i(v, G - e) = \mu_i(v, G - e) = 1$ for every $e \in E(G)$ which implies v is κ_i and μ_i-stable.

With one initial condition we obtain a characterization of a μ_i-stable vertex. In order to make the statement of the next theorem easier, we define a "dominant" neighbor for a vertex v. Let x be a neighbor of v so that, given any μ_i-set for v, S_v, there is no other neighbor of v in the component of G - S_v - v containing x. Then x is called "dominant".

Theorem 4.11: Let v $\in$ V(G) be such that v has no dominant neighbor. Then v is μ_i-stable if and only if no μ_i-set for v in G contains an edge.

Proof: Let v $\in$ V(G) be such that v has no neighbor which gets separated from all the other neighbors of v in G - S_m - v for every μ_i-set S_m for v in G.

Suppose v has a μ_i-set S_m containing an edge e. Then μ_i(v, G - e) $\leq$ | S_m | - 1 since v is a cutvertex in (G - e) - (S_m - e) = G - S_m. Thus μ_i(v, G - e) < | S_m | = μ_i(v, G) and v is not μ_i-stable.

Suppose no μ_i-set for v in G contains an edge. Note that deg(v) $\neq$ 1, otherwise v would have a μ_i-set consisting entirely of edges. By way of contradiction, let e be an edge of G such that μ_i(v, G) $\neq$ μ_i(v, G - e).

Case 1: Suppose e is not incident with v. Since μ_i(v, G) $\neq$ μ_i(v, G - e), Theorem 4.2 implies that μ_i(v, G - e) = μ_i(v, G) - 1. Let S_m* be a μ_i-set for v in G - e and let u and w be neighbors of v in G - e which are in different components of (G - e) - S_m* - v (such vertices exist by the hypothesis of the theorem). Since (G - e) - S_m* = G - (S_m* $\cup$ { e }) and v is a cutvertex in (G - e) - S_m*, then (S_m* $\cup$ { e }) is a μ_i-set for v in G, contradicting the supposition that no μ_i-set for v in G contains an edge and thus this case does not occur.

Case 2: Suppose e is incident to v. Since μ_i(v, G) $\neq$ μ_i(v, G - e), Theorem 4.2 implies that μ_i(v, G - e) > μ_i(v, G). Let e = zv. Then by the hypothesis of the theorem, there exist vertices u and w distinct from z where u and w are neighbors of v which are separated in G - S_m - v, for some μ_i-set S_m for v in G. Thus there are a maximum of μ_i(v, G) internally disjoint paths between u and w in G - v. But there are

also a maximum of $\mu_i(v, G)$ internally disjoint paths between u and w in (G - e) - v. This implies that $\mu_i(v, G - e) \leq \mu_i(v, G)$ (with equality since e is incident with v), a contradiction.

Thus, there does not exist an edge e such that $\mu_i(v, G) \neq \mu_i(v, G)$ which implies v is μ_i-stable. ❑

The graph in Figure 4.9 demonstrates that the initial condition of the preceding theorem is necessary.

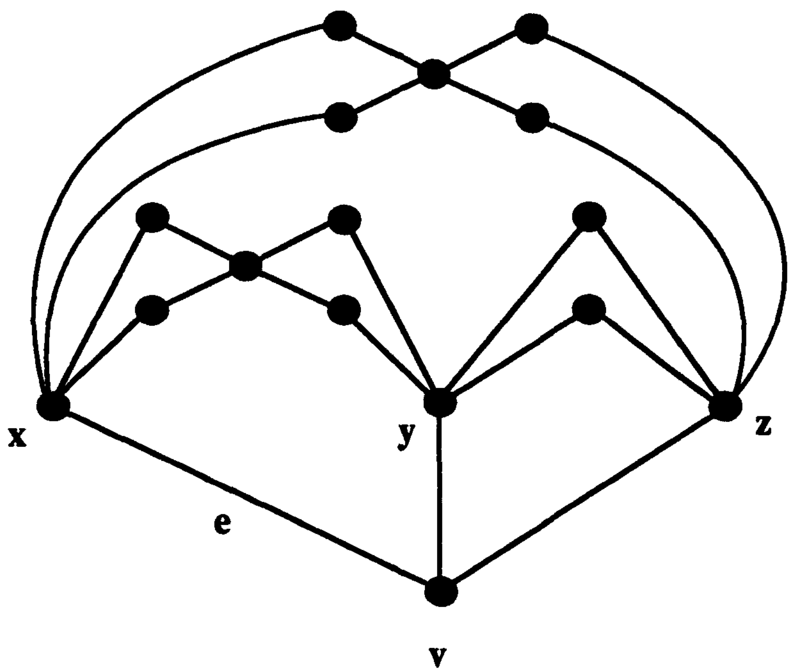


Figure 4.9 A graph illustrating the initial condition in Theorem 4.11.

In this graph, $\mu_i(v, G) = 2$ with the separation of neighbors x and y, or x and z with each μ_i-set consisting of two vertices. Note that *the* vertex x is *dominant*. In G - e there exist a maximum of three internally disjoint paths between y and z which implies $\mu_i(v, G - e) = 3$ and v is not μ_i-stable in G.

When a vertex is μ_i-stable, we formalize the relationship between the κ_i and μ_i values for that vertex.

Corollary 4.12: If $v \in V(G)$ is μ_i-stable, then $\kappa_i(v) = \mu_i(v)$ and further, every μ_i-set for v in G is a κ_i-set for v in G and vice versa.

Proof: If $v \in V(G)$ is μ_i-stable, then no μ_i-set for v in G contains an edge and so any μ_i-set is a κ_i-set. This further implies that $\mu_i(v) = \kappa_i(v)$, making every κ_i-set a μ_i-set for v in G. ❑

The contrapositive of the previous corollary is extended to include κ_i-stability.

Theorem 4.13: If $\mu_i(v) < \kappa_i(v)$ for $v \in V(G)$, then v is neither κ_i-stable nor μ_i-stable.

Proof: Let $v \in V(G)$ where $\mu_i(v) < \kappa_i(v)$. By Theorem 3.19, there exists a μ_i-set S_m for v in G such that S_m contains exactly one edge, e = uw, with u and w neighbors of v. So there are a maximum of $| S_m |$ internally disjoint uw paths in G. But there are a maximum of $| S_m | - 1$ internally disjoint uw paths in G e (u and w remain neighbors in G - e but are no longer adjacent). Hence v is not μ_i-stable.

Note that $\deg_G(v) \neq 1$ since if $\deg(v) = 1$, then $\mu_i(v) = \kappa_i(v)$, a contradiction. Also $< N_{G-e}(v) >$ is clearly not complete, so by Theorem 3.3, we have $\kappa_i(v, G - e) \leq | S_m - e | = \mu_i(v, G) - 1 < \mu_i(v, G) < \kappa_i(v, G)$. Therefore, v is not κ_i-stable. ❑

In the example in Figure 4.9, there were no edges in any μ_i-set S_m for v in G. Note also that the maximum number of edge disjoint paths is more than the maximum number of internally disjoint paths between any pair of neighbors of v that are separated in $G - S_m - v$. This relationship is explored in Theorem 4.16, after a result by Boland [2] and a corresponding corollary are presented.

Theorem 4.14: For any $v \in V(G)$, $\mu_i(v) < \lambda_i(v)$ if and only if every μ_i-set for v contains a vertex.

An immediate result from Theorem 4.14 is Corollary 4.15.

Corollary 4.15: If there is no edge in any μ_i-set for v in G then $\mu_i(v) < \lambda_i(v)$.

Theorem 4.16: Suppose G is a graph with v a vertex of G which has no edges in any μ_i-set for v and $\mu_i(v) > 0$. Let S_m be a particular μ_i-set which separates two neighbors of v, u and w, in G - S_m - v. Then the maximum number of edge disjoint u-w paths in G is greater than the maximum number of internally disjoint u-w paths in G.

Proof: Let $v \in V(G)$ where there is no edge in any μ_i-set for v in G. Let S_m be an arbitrary μ_i-set for v in G where the two neighbors of v separated in G - S_m - v are u and w. By Corollary 4.15, the maximum number of internally disjoint u-w paths, $\mu_i(v)$, is strictly less than the maximum number of edge disjoint u-w paths, $\lambda_i(v)$. ❑

The converse of the previous theorem is not true as evidenced by the counterexample in Figure 4.10. In G - v there exist a maximum of three edge disjoint u-w paths and a maximum of two internally disjoint u-w paths. But { x, uw } is a μ_i-set which, of course, contains an edge.

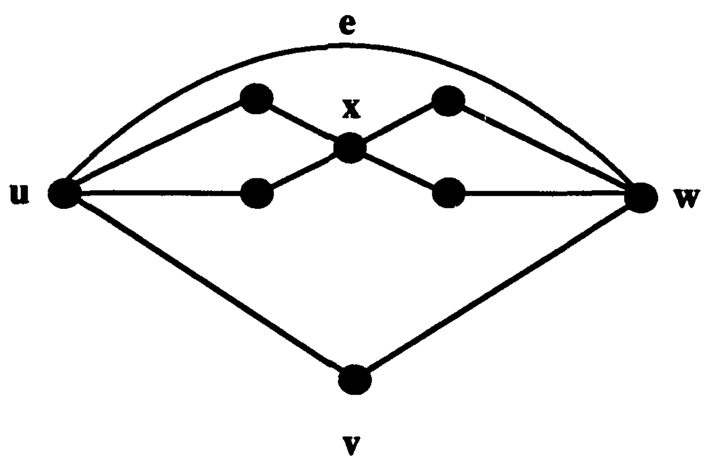


Figure 4.10 A counterexample to the converse of Theorem 4.16.

The idea of the "dominant" neighbor involved in Theorem 4.11 is vital when investigating the μ_i-stability of a vertex. In fact, the mere presence of a "dominant" neighbor will prevent μ_i-stability.

Theorem 4.17: If there is a dominant neighbor of $v \in V(G)$, then v is not μ_i-stable in G.

Proof: Let u be a dominant neighbor of v and let e = uv. Then μ_i must change when e is removed, since u is no longer a neighbor. Thus, v is not μ_i-stable in G. ❑

The converse of Theorem 4.17 is easily disproved using the counterexample in Figure 4.11. Here, vertex v is not μ_i-stable since $\mu_i(v, G) = 2$ and $\mu_i(v, G - e) = 1$. But there are disjoint pairs of neighbors separated by μ_i-sets. For example, w and y are separated in $G - S_m - v$ where $S_m = \{ u, x \}$, and u and x are separated in $G - S_m^* - v$ where $S_m^* = \{ w, y \}$.

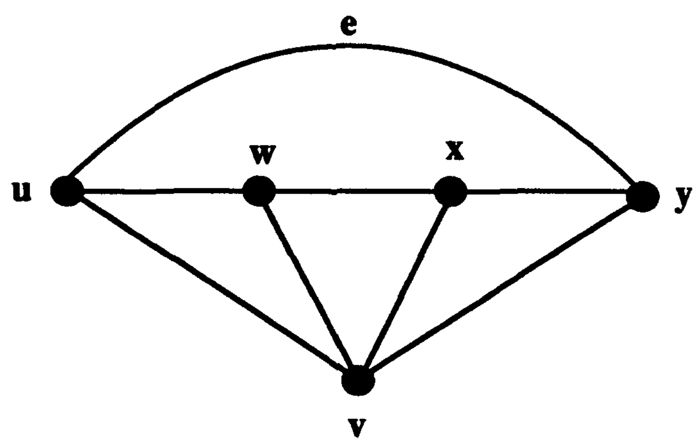


Figure 4.11 A counterexample to the converse of Theorem 4.17.

We now examine the primary difference between a κ_i-set and a μ_i-set; that is, the case where a κ_i-set isolates a vertex in a K_2 component.

Theorem 4.18: For $v \in V(G)$, if *any* κ_i-set for v in G isolates v in a K_2 component, then v is not κ_i-stable.

Proof: Let S_v be a κ_i-set for v in G that isolates v in a K_2 component with vertex u. Since $G - S_v - v$ has a neighbor of v as an isolated vertex, then S_v is a neighborhood κ_i-set. Thus if $e = uw$ where $w \neq v$, then in $G - e$ the removal of the set of vertices $\{ S_v - w \}$ will make v a cutvertex since v and u will be isolated in a K_2 component in $(G - e) - (S_v - w)$. Therefore $\kappa_i(v, G - e) \leq | S_v - w | = \kappa_i(v, G) - 1 < \kappa_i(v, G)$ implying v is not κ_i-stable. ❑

The previous argument can be immediately extended to include neighborhood i-connectivity sets for μ_i as well. The logic is identical and hence the proof has been omitted.

Theorem 4.19: For $v \in V(G)$, if there exists a neighborhood κ_i-set (μ_i-set) for v in G, then v is not κ_i- (μ_i) stable.

We now arrive at the most important result of this chapter, the relationship between κ_i-stability and μ_i-stability under edge deletion, presented in Theorem 4.21. First we present a lemma.

Lemma 4.20: If $v \in V(G)$ is μ_i-stable, then for $e \in E(G)$, there exists a set of elements that is a μ_i-set for v in G *and* G - e.

Proof: Let $v \in V(G)$ be μ_i-stable and $e \in E(G)$.

Case 1: Let e be incident with v. Let $S_m{*}$ be any μ_i-set for v in G - e. So $| S_m{*} | = \mu_i(v, G - e) = \mu_i(v\ G)$. Since $(G - e) - v = G - v$ then $S_m{*}$ separates the same neighbors of v in $G - S_m{*} - v$ and has the necessary cardinality, so $S_m{*}$ is a μ_i-set for v in G.

Case 2: Suppose e is not incident with v and let S_m be any μ_i-set for v in G. Since $N_G(v) = N_{G-e}(v)$, the same neighbors of v will be separated in $G - S_m - v$ as in

$(G - e) - S_m - v$. Since v is μ_i-stable, $| S_m | = \mu_i(v, G) = \mu_i(v, G - e)$ which implies S_m is a μ_i-set for v in G - e. ❑

Theorem 4.21: Vertex $v \in V(G)$ is κ_i-stable under edge deletion if and only if v is μ_i-stable under edge deletion.

Proof: Let $v \in V(G)$ be μ_i-stable. Then by Corollary 4.12, $\kappa_i(v, G) = \mu_i(v, G)$ and every μ_i-set for v in G is a κ_i-set for v in G. Let e be an arbitrary edge from E(G). From Lemma 4.20, there exists a μ_i-set S_m for v in G and G - e. Note that S_m consists entirely of vertices.

Thus we have a set of vertices in G - e whose removal makes v a cutvertex. So $\kappa_i(v, G - e) \leq | S_m | = \mu_i(v, G - e)$ which implies $\kappa_i(v, G - e) = \mu_i(v, G - e) = \mu_i(v, G) = \kappa_i(v, G)$. Since e was arbitrary, v is κ_i-stable.

Now let $v \in V(G)$ where v is not μ_i-stable. We will show v is not κ_i-stable. Then there exists an edge $e \in E(G)$ where $\mu_i(v, G) \neq \mu_i(v, G - e)$. Note that if $deg_G(v) = 1$, v is not κ_i-stable and we are done, so let $deg_G(v) \geq 2$.

Case 1: Suppose $\mu_i(v, G) < \kappa_i(v, G)$. Then by Theorem 4.13, v is not κ_i-stable.

Case 2: Suppose $\mu_i(v, G) = \kappa_i(v, G)$.

Case 2a: Suppose $\mu_i(v, G - e) > \mu_i(v, G)$. Then $\kappa_i(v, G - e) \geq \mu_i(v, G - e) > \mu_i(v, G) = \kappa_i(v, G)$ which implies v is not κ_i-stable.

Case 2b: Suppose $\mu_i(v, G - e) < \mu_i(v, G)$. If the κ_i value for v changes from G to G - e we are done, so assume $\mu_i(v, G - e) < \kappa_i(v, G) = \kappa_i(v, G - e)$. So by Theorem 3.19, there exists a μ_i-set $S_m{*}$ for v in G - e with exactly one edge and that edge is between neighbors of v. By Theorem 3.20, $(G - e) - S_m{*} - v$ has exactly two components, C_1 and C_2, which contain vertices of $N_{G-e}(v)$, so assume $u \in V(C_1)$ and $w \in V(C_2)$ with $e' = uw \in S_m{*}$.

So there are $| S_m{*} | = \mu_i(v, G - e)$ internally disjoint u-w paths in (G - e) - v. Since the μ_i value decreased, the edge e is not incident with v. Then in G - v, there can be at most $| S_m{*} | + 1$ internally disjoint uw paths. Since e' = uw is itself one of

the uw paths, then in G - e' - v we can have at most | S_m* | internally disjoint uw paths.

Since u and w are not adjacent in G - e', then by Theorem 3.3, κ_i(v, G - e') ≤ | S_m* | since deg(v) ≠ 1 and < $N_{G-e'}$(v) > is not complete. So κ_i(v, G - e') ≤ | S_m* | = μ_i(v, G - e) < κ_i(v, G - e) = κ_i(v, G). Thus there exists an edge in G (namely e' = uw) whose removal causes the κ_i value for v to decrease which implies v is not κ_i-stable. ❑

The reason that μ_i-stability implies κ_i-stability for edge deletion and not edge addition is that μ_i-stability for edge deletion implies every μ_i-set consists entirely of vertices. In edge addition, a μ_i-set can include an edge, particularly the edge between the two separated neighbors which prevents them from being separated by a κ_i-set.

CHAPTER 5

NEUTRAL EDGES AND STABLE GRAPHS FOR INCLUSIVE CONNECTIVITY

Introduction

We begin our study of neutral edges in this chapter. Recall from the definition of neutral edge in Chapter 2, the removal of a neutral edge does not change the respective i-connectivity value of any vertex.

We first explore the various possible combinations of inclusive connectivity neutrality for an edge. Examples for each possible combination will be provided.

Next we examine what changes with respect to another edge can occur in a graph upon the deletion of a neutral edge. A surprising result is included on how the total number of λ_i-neutral edges in a graph can change upon the deletion of a λ_i-neutral edge.

Finally, we investigate stable graphs, with respect to the "sum-stable" definition from [20] for various combinations of inclusive connectivity. The possibility that a graph may have *every edge* κ_i-neutral is explored, with an extremely interesting result.

Combinations of Inclusive Connectivity Neutrality

As stated previously, the term "λ_i-neutral" for an edge is identical to "stable edge" found in [19]. The first result concerning λ_i-neutrality was established by Rice [18] and is presented next.

Theorem 5.1: If an edge $e \in E(G)$ is in a λ_i-set for some vertex $v \in V(G)$, then e is not λ_i-neutral.

Simply put, if an edge e is in a λ_i-set for v in G, then you can delete that edge and the λ_i value for v in G - e is one less then the value in G. This idea is extended to μ_i in the next theorem.

Theorem 5.2: If an edge $e \in E(G)$ is in a μ_i-set for some vertex $v \in V(G)$, then e is not μ_i-neutral.

Proof: If $e \in E(G)$ is in some μ_i-set S_m for v in G, then $\mu_i(v, G) = | S_m |$. So $\mu_i(v, G - e) = | S_m | - 1$ since $G - S_m - v = (G - e) - (S_m - e) - v$. Therefore, e is not μ_i-neutral since $\mu_i(v, G - e) < \mu_i(v, G)$. ❑

Since there are no edges in a κ_i-set, a similar result for κ_i is not possible. In fact, an edge incident to a vertex that is in some κ_i-set for v in G may or may not possess κ_i-neutrality. In the graph of Figure 2.7, edge e is incident to a vertex that is in a κ_i-set for the four vertices of degree two, but e is κ_i-neutral. And in Figure 4.2, again e is incident to a vertex in a κ_i-set for v, except $\kappa_i(v, G) = 2$ while $\kappa_i(v, G - e) = 1$ implying e is not κ_i-neutral.

Surprisingly, if an edge is λ_i-neutral, then that edge may still be present in a minimum edge disconnecting set. This situation is illustrated in the graph of Figure 5.1.

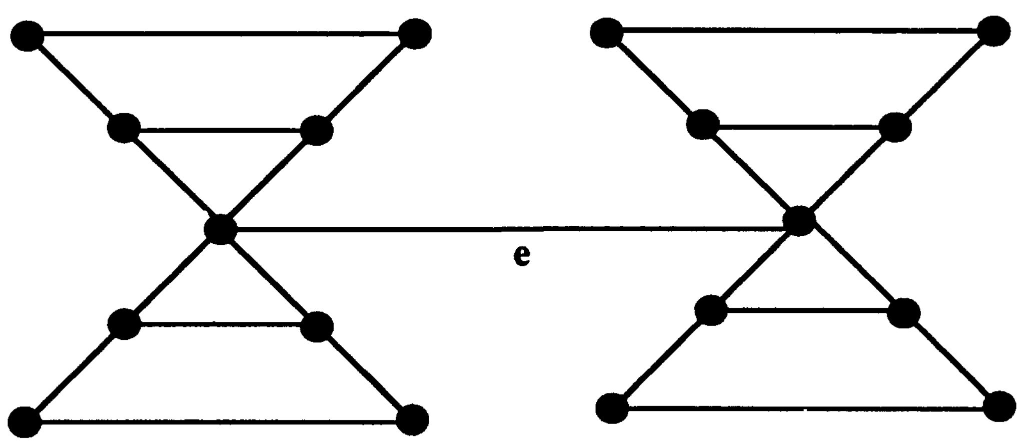


Figure 5.1 A λ_i-neutral edge that is in a minimum edge disconnecting set.

Here, the edge e is λ_i-neutral since $\lambda_i(v, G) = \lambda_i(v, G - e) = 1$ for all vertices except the endpoints of e which are cutvertices in G and G - e. But e is clearly in a minimum edge disconnecting set for G since $\lambda(G) = 1$.

But an edge e being λ_i-neutral does imply, for every $v \in V(G)$, the existence of a set of $\lambda_i(v)$ edge disjoint paths in G - v between a set of neighbors of v where e is not on any of these paths. This idea is formalized in Theorem 5.3.

Theorem 5.3: If $e \in E(G)$ is λ_i-neutral then for all $v \in V(G)$, where $\deg(v) > 1$, there exists a set of $\lambda_i(v)$ edge disjoint paths between a set of neighbors of v, where e is not on any of these paths.

Proof: Let $e \in E(G)$ be λ_i-neutral. Let v be an arbitrary vertex of G and to insure there exists a pair of neighbors of v, let $\deg(v) \geq 2$. Given a pair of neighbors of v separated by a maximum of $\lambda_i(v)$ edge disjoint paths in G - v, suppose e is on one path. Further, suppose u and w are a pair of neighbors of v separated in this manner. Then we construct a λ_i-set S_e for v as follows:

(a) Place edge e in S_e.

(b) Since there are $\lambda_i(v) - 1$ edge disjoint u-w paths in G - v - e, take any set of $\lambda_i(v)$ -1 edges that will separate u and w in G - v - e, and place these edges in S_e.

Thus $u, w \in N(v)$ will be separated in $G - S_e - v$ where $e \in S_e$. Since e is contained in a λ_i-set, then by Theorem 5.1 e is not λ_i-neutral, a contradiction. Therefore, there must exist at least one set of $\lambda_i(v)$ edge disjoint u-w paths that does not include e. ❑

The graph in Figure 5.2 provides a counterexample to the converse of Theorem 5.3. Here, $\lambda_i(v, G) = 1$ and there exists a u-w path in G - v that does not include e. But e is not λ_i-neutral since $\lambda_i(u, G) = 1$ but $\lambda_i(u, G - e) = 0$.

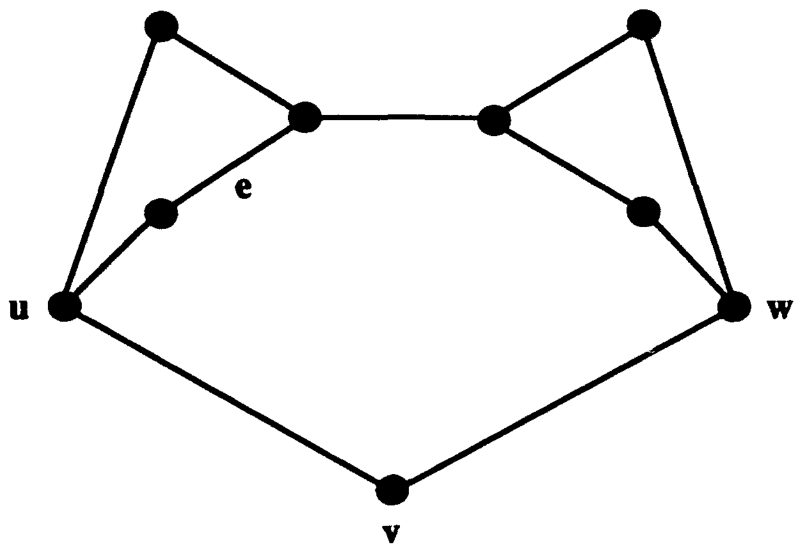


Figure 5.2 A counterexample to the converse of Theorem 5.3.

Now we explore whether every possible combination of neutrality among the three i-connectivities is realizable. For example, the edge e in the graph of Figure 5.1 is λ_i-neutral, κ_i-neutral, and μ_i-neutral. Hence, we prove Theorem 5.4.

Theorem 5.4: Each of the eight combinations of neutrality for an edge among the inclusive connectivity parameters has an infinite class of graphs satisfying it.

Proof:

Case (1): An edge that is not λ_i-neutral, κ_i-neutral, or μ_i-neutral.

Any edge that is on one of the paths from every maximum set of internally (edge) disjoint paths between separated neighbors associated with a κ_i and μ_i- (λ_i) set for a vertex will not be κ_i and μ_i- (λ_i-) neutral. The graph of Figure 5.3 gives an infinite class of such graphs with $n \geq 2$, where $\lambda_i(v, G) = \kappa_i(v, G) = \mu_i(v, G) = 2$, but $\lambda_i(v, G - e) = \kappa_i(v, G - e) = \mu_i(v, G - e) = 1$.

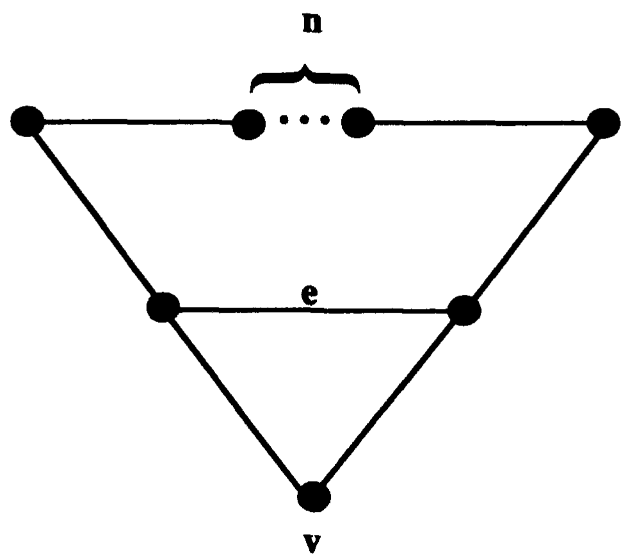


Figure 5.3 An edge that is not λ_i, κ_i, or μ_i-neutral.

Case (2): An edge that is λ_i-neutral, κ_i-neutral, and μ_i-neutral.

An edge e that is λ_i-neutral, κ_i-neutral, and μ_i-neutral is displayed in the graph of Figure 5.4. For $n \geq 2$, *this infinite class of graphs has* $\lambda_i(v, G) = \kappa_i(v, G) = \mu_i(v, G) = 1$ for every $v \in V(G)$. And in $G - e$ we have $\lambda_i(v, G - e) = \kappa_i(v, G - e) = \mu_i(v, G - e) = 1$ to establish neutrality for all three i-connectivity parameters, again for any vertex.

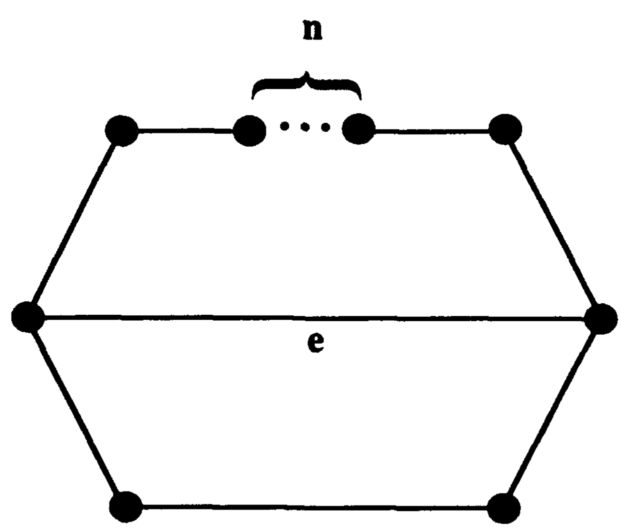


Figure 5.4 An edge that is λ_i, κ_i, and μ_i-neutral.

Case (3): An edge that is λ_i-neutral and κ_i-neutral but not μ_i-neutral.

In Figure 5.5, $\lambda_i(w, G) = \lambda_i(w, G - e) = \kappa_i(w, G) = \kappa_i(w, G - e)$ for all $w \in V(G)$, where each value is either 1, 3 or 5 depending upon which vertex w is chosen. Now $\mu_i(w, G) = \mu_i(w, G - e)$ for all $w \in V(G)$ except for w = v or u. In these cases, $\mu_i(v, G) = \mu_i(u, G) = 3$, but $\mu_i(v, G - e) = 4$ and $\mu_i(u, G - e) = 2$. The reason that the μ_i value for v changes is that the other endpoint of e is involved in the unique pair of neighbors of v that produce the μ_i value. The μ_i value for u changes since e cuts one of the internally disjoint paths between its neighbors. Thus edge e is λ_i-neutral and κ_i-neutral but not μ_i-neutral.

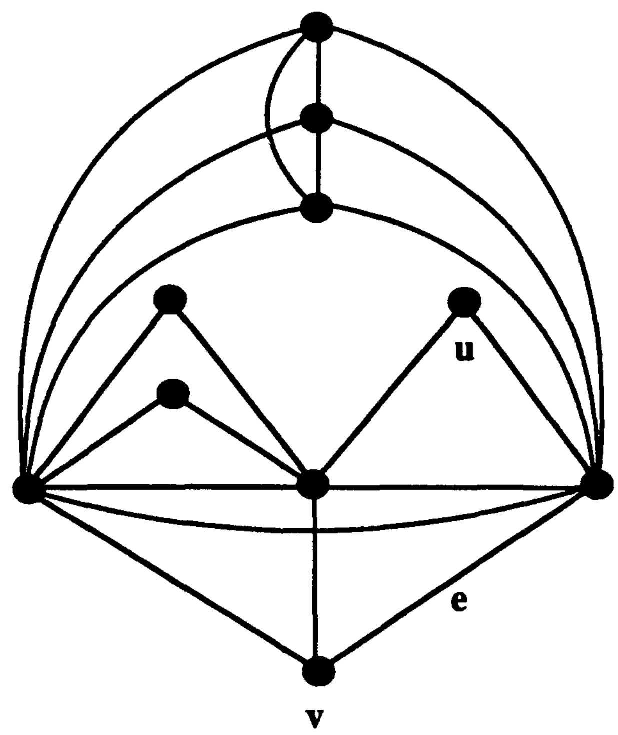


Figure 5.5 An edge that is λ_i and κ_i-neutral, but not μ_i-neutral.

An infinite class of graphs with such an edge is shown in Figure 5.6 for $n \geq 1$. Again, the only vertices whose μ_i-value changes are vertex v and the group of n

vertices of degree two. These values are $\mu_i(v, G) = n + 2$ and $\mu_i(v, G - e) = n + 3$, while $\mu_i(u, G) = n + 2$, and $\mu_i(u, G - e) = n + 1$.

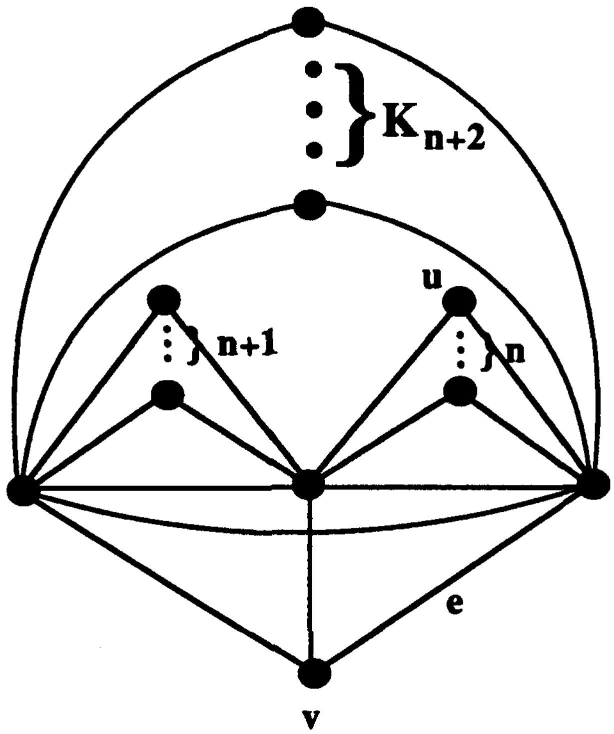


Figure 5.6 An infinite class of graphs for Case (3).

Case (4): An edge that is λ_i-neutral, but not κ_i or μ_i-neutral.

The λ_i value for every vertex in the graph in Figure 5.7 remains the same between G and G - e, while the κ_i and μ_i values in G and G - e remain the same for every vertex except v. For this one exception, $\kappa_i(v, G) = \mu_i(v, G) = 2$, but $\kappa_i(v, G - e) = \mu_i(v, G - e) = 1$. There are a maximum of two internally disjoint u-w paths in G, but a κ_i and μ_i-set for v in G - e is { z }. There are a maximum of three *edge* disjoint u-w paths in G and G - e, which preserves λ_i-neutrality for v. Thus e is λ_i-neutral, but not κ_i or μ_i-neutral.

The edge xy can be subdivided indefinitely to provide an infinite class of graphs having this type of edge e, since all i-connectivity values for x and y in G and G - e equal one.

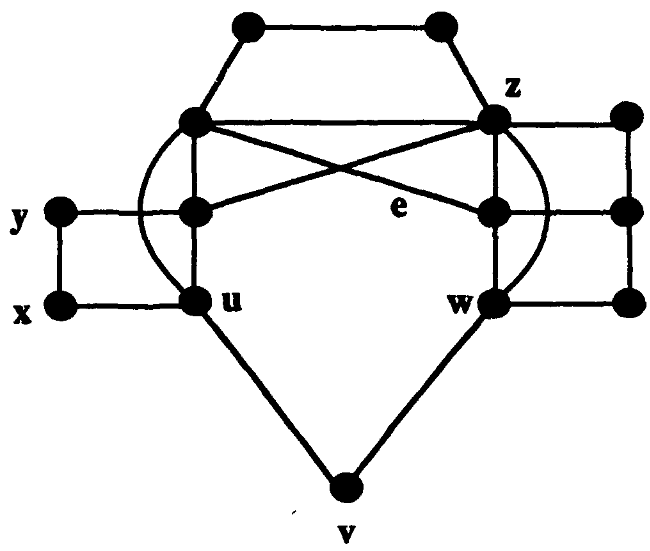


Figure 5.7 An edge that is λ_i-neutral, but not κ_i or μ_i-neutral.

Case (5): An edge that is κ_i-neutral, but not λ_i or μ_i-neutral.

In this case, the i-connectivity values for every vertex except v in the graph of Figure 5.8 remain the same for G and G - e. For v, $\kappa_i(v, G) = \kappa_i(v, G - e) = 4$, but the λ_i and μ_i values decrease from 4 to 3 and 3 to 2, respectively, after the removal of e. A λ_i-set for v in G - e is { xy, yz, uw } separating neighbors y and z. A μ_i-set for v in G - e is { z, xy } separating neighbors x and y.

As in case (4), to obtain an infinite class of such graphs with an edge e of this type the edge ab can be subdivided indefinitely, since all i-connectivity values for a and b in G and G - e equal one.

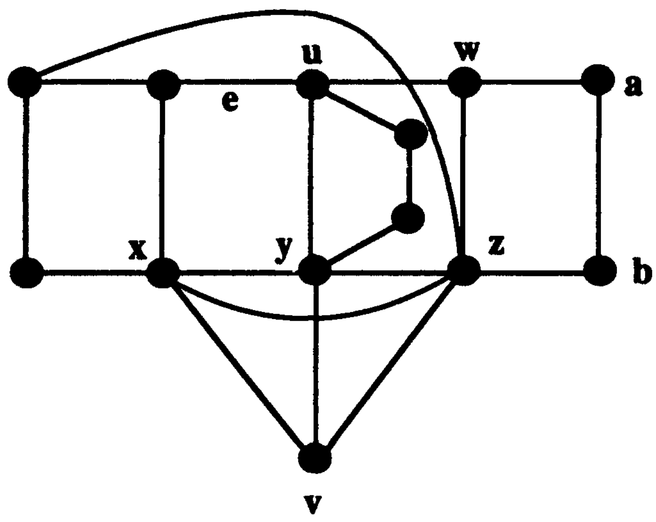


Figure 5.8 An edge that is κ_i-neutral, but not λ_i or μ_i-neutral.

Case (6): An edge that is κ_i and μ_i-neutral, but not λ_i-neutral.

The i-connectivity values for all vertices except x and v in the graph of Figure 5.9 are equal to one in G and G - e. The i-connectivity values for x are all equal to two in G and G - e while $\kappa_i(v, G) = \mu_i(v, G) = \kappa_i(v, G - e) = \mu_i(v, G - e) = 1$, but $\lambda_i(v, G) = 2$ and $\lambda_i(v, G - e) = 1$.

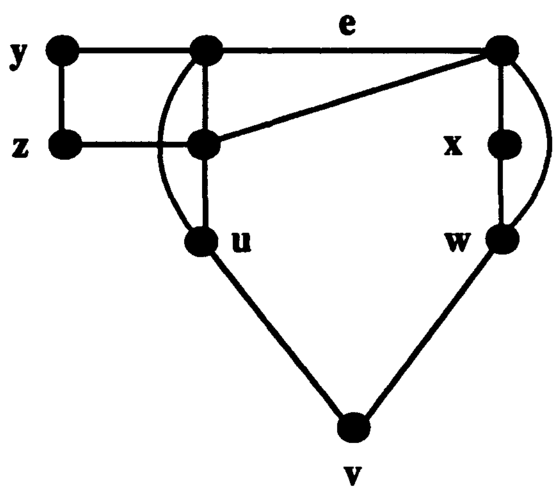


Figure 5.9 An edge that is κ_i and μ_i-neutral, but not λ_i-neutral.

Indefinitely subdividing the edge yz produces an infinite class of graphs with an edge that is κ_i and μ_i-neutral, but not λ_i-neutral.

Case (7): An edge that is μ_i-neutral, but not κ_i or λ_i-neutral.

All vertices in the graph of Figure 5.10 except v, a, and b have i-connectivity values in G and G - e equal to one. Vertices a and b have i-connectivity values in G and G - e equal to two. Vertex v provides the required changes as $\mu_i(v, G) = \mu_i(v, G - e) = 2$, $\kappa_i(v, G) = \lambda_i(v, G) = 3$, and $\kappa_i(v, G - e) = \lambda_i(v, G - e) = 2$. With the removal of e, the cardinality of the neighborhood κ_i and λ_i-sets for v decreases by one.

Again, we can repeatedly subdivide the edge xy to obtain an infinite class of graphs with this type of edge.

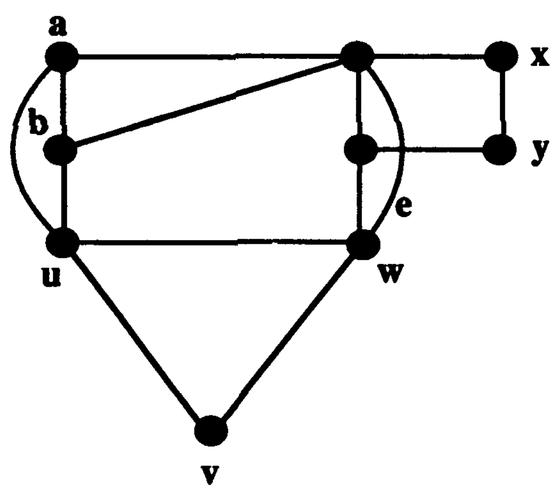


Figure 5.10 An edge that is μ_i-neutral, but not λ_i or κ_i-neutral.

Case (8): An edge that is μ_i and λ_i-neutral, but not κ_i-neutral.

Since the neighborhood of v in the graph of Figure 5.11 is complete, $\kappa_i(v, G) = 4$. But $\kappa_i(v, G - e) = 3$ with a κ_i-set being { b, u, w }. Note that the λ_i values for v in G and G - e are four while the μ_i values for v in G and G - e are three. For the remaining

vertices, all i-connectivity values are equal to one except for vertices x and y which have values equal to four.

The edge to indefinitely subdivide in this case to achieve an infinite class is edge ab.

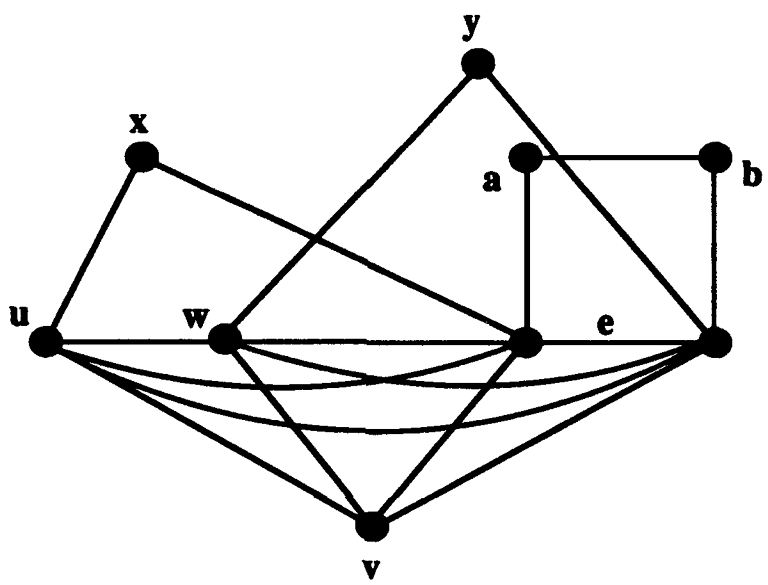


Figure 5.11 An edge that is μ_i and λ_i-neutral, but not κ_i-neutral.

This case completes the proof. ❑

It should be noted that an important part of the examples of Theorem 5.4 is the existence of a vertex of degree two. The advantage of using vertices of degree two is that any vertex adjacent to a degree two vertex will have i-connectivity values equal to one (assuming no cutvertices are in the graph). Since many of the vertices in these examples are of degree two (often utilized in pairs), many of the i-connectivity values are one, simplifying the verification process. The remaining problem is to insure that

those vertices whose i-connectivity values are not equal to one will not be affected with the removal of the specified edge, except for the desired changes.

Deletion of a Neutral Edge

Upon the deletion of a neutral edge, it is interesting to note any changes that may occur in the graph. In particular, it is natural to observe what happens to a neutral edge after the deletion of another neutral edge.

It is possible that the deletion of a neutral edge from a graph will have no effect upon the neutrality of another neutral edge. For the graph in Figure 5.12, all vertices have i-connectivity values equal to one for all three parameters for G, G - e_1, G - e_2, and G - e_1 - e_2. Thus e_1 and e_2 remain λ_i, κ_i, and μ_i-neutral in G - e_2 and G - e_1, respectively. Therefore we have an example of a λ_i- (κ_i, μ_i) neutral edge that remains λ_i- (κ_i, μ_i) neutral upon the deletion of another λ_i- (κ_i, μ_i) neutral edge.

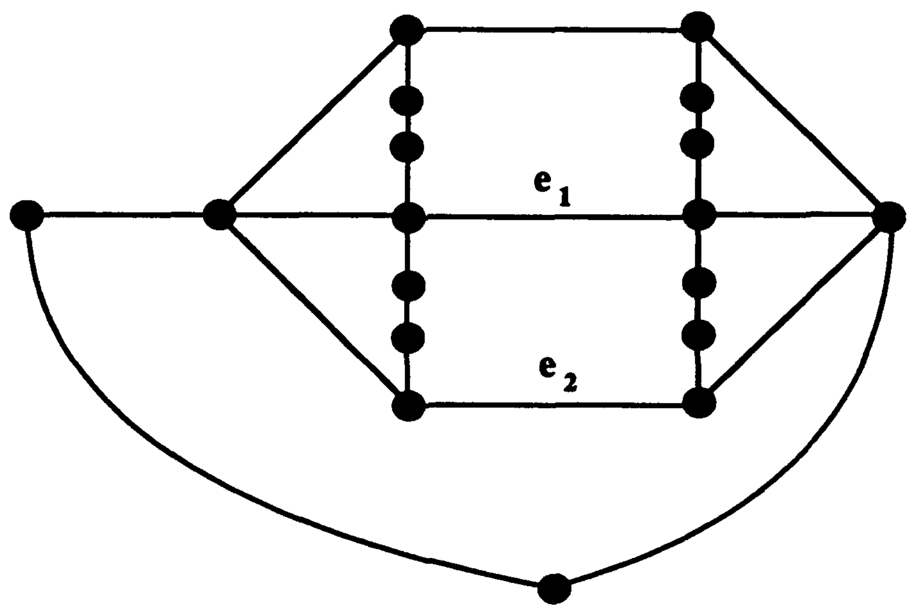


Figure 5.12 An edge whose neutrality is preserved upon removal of a neutral edge.

But it is also possible for an edge to lose its neutrality with the deletion of another neutral edge. In the graph of Figure 5.13, $\lambda_i(v, G) = \kappa_i(v, G) = \mu_i(v, G) = 1$ for all $v \in V(G)$. All inclusive connectivity values remain one in $G - e_1$ and $G - e_2$ establishing the λ_i, κ_i, and μ_i-neutrality of e_1 and e_2. But in $G - e_1 - e_2$, vertices u, v, x, and y become cutvertices implying e_1 is not λ_i, κ_i, or μ_i-neutral in $G - e_2$ and e_2 is not λ_i, κ_i, or μ_i-neutral in $G - e_1$. Thus we have two examples of a λ_i- (κ_i, μ_i) neutral edge that do not remain λ_i- (κ_i, μ_i) neutral upon the deletion of another λ_i- (κ_i, μ_i) neutral edge.

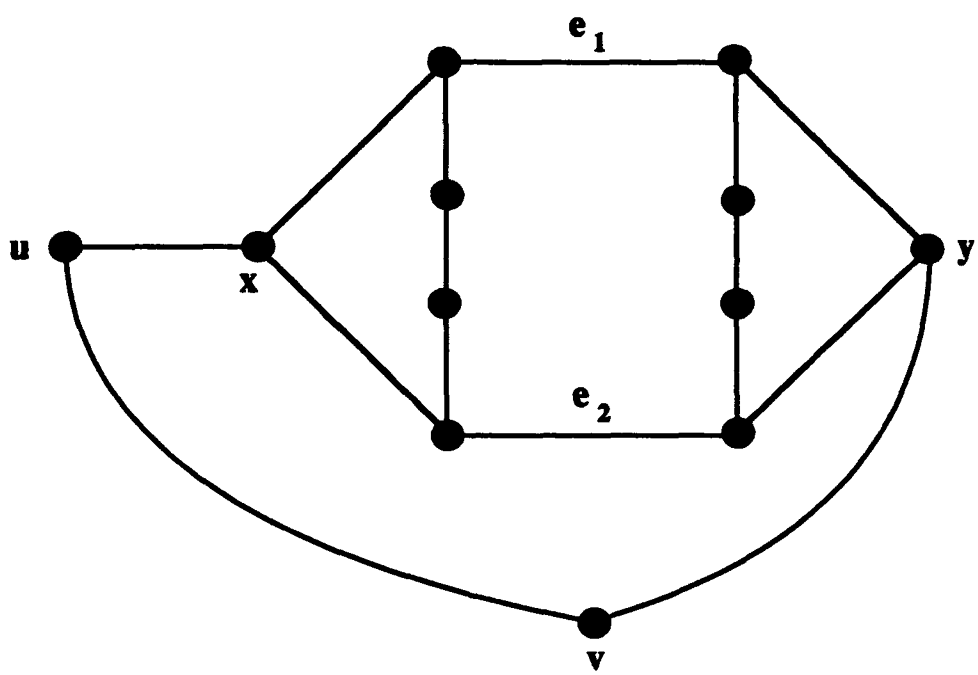


Figure 5.13 An edge whose neutrality is not preserved.

The most interesting case in the deletion of a neutral edge is the possibility of a non-neutral edge becoming neutral. That an edge that is not neutral can become neutral with the deletion of an edge that does not affect the i-connectivity value of any

vertex in the graph is certainly not intuitive. With the assumption that this case is indeed possible, we prove the following theorem concerning the location of those edges in question.

Theorem 5.5: If e_1 is λ_i-neutral and e_2 is not λ_i-neutral in G but e_2 is λ_i-neutral in G - e_1, then there exists a vertex $v \in V(G)$ such that $\lambda_i(v, G) = \lambda_i(v, G - e_1 - e_2) \neq \lambda_i(v, G - e_2)$ and exactly one of e_1 and e_2 is incident with v.

Proof: If e_1 is λ_i-neutral in G and e_2 is λ_i-neutral in G - e_1, then $\lambda_i(v, G) = \lambda_i(v, G - e_1) = \lambda_i(v, G - e_1 - e_2)$ for every $v \in V(G)$. In addition, if e_2 is not λ_i-neutral in G, then $\lambda_i(v, G) \neq \lambda_i(v, G - e_2) \neq \lambda_i(v, G - e_1 - e_2)$. So by Theorem 4.1, we have two possible cases:

(1) $\lambda_i(v, G) = \lambda_i(v, G - e_2) + 1 = \lambda_i(v, G - e_1 - e_2)$

(2) $\lambda_i(v, G) = \lambda_i(v, G - e_2) - 1 = \lambda_i(v, G - e_1 - e_2)$

In (1) e_2 must not be incident with v, while e_1 must be incident with v. The opposite occurs in (2) where e_1 must not be incident with v while e_2 is and the conclusion follows. ❑

Upon further consideration Theorem 5.5 gives the two cases when an edge that is not λ_i-neutral becomes λ_i-neutral after the deletion of a λ_i-neutral edge: the λ_i value for a vertex can increase by one and then decrease by one, or vice versa. Through the direct application of Theorem 4.1, Theorem 5.5 also specifies the location of the two edges in question, i.e., the edge being deleted or the edge whose neutrality changes will be incident with that vertex while the other will not.

The existence of such a neutrality change is proven in the graph of Figure 5.14. For every $c \in V(G)$ where $c \neq v$, $\lambda_i(c, G) = \lambda_i(c, G - e_1) = \lambda_i(c, G - e_2) = \lambda_i(c, G - e_1 - e_2) = 1$. But $\lambda_i(v, G) = \lambda_i(v, G - e_1) = \lambda_i(v, G - e_1 - e_2) = 4$ and $\lambda_i(v, G - e_2) = 5$. There exist a maximum of four edge disjoint xy and xz paths in G - v where one λ_i-set is the neighborhood λ_i-set from vertex x. But there exist a maximum of five edge disjoint yz paths in G - v, which implies the λ_i value for v must increase by one upon

the removal of e_2 The removal of e_1 from G - e_2 will cut one of the edge disjoint yz paths in G - e_2, thus decreasing the λ_i value for v to its original value in G. This implies that e_2 is not λ_i-neutral in G.

If the order of the deletion of the two edges is reversed, then we can establish the λ_i-neutrality of e_2 in G - e_1. Combining the fact that x remains a neighbor of v in G - e_1 and that there are still a maximum of four edge disjoint xy paths in G - e_1 - v with Theorem 4.1, we have $\lambda_i(v, G - e_1) = 4$. Now the removal of e_2 from G - e_1 must keep the λ_i value for v constant since we already have $\lambda_i(v, G - e_1 - e_2) = 4$. Thus, e_2 becomes λ_i-neutral in G - e_1, providing the "nonintuitive" example we sought!

Note that Theorem 5.5 is clearly demonstrated in the graph of Figure 5.14, with the required proximity of e_1 and e_2 to the vertex v.

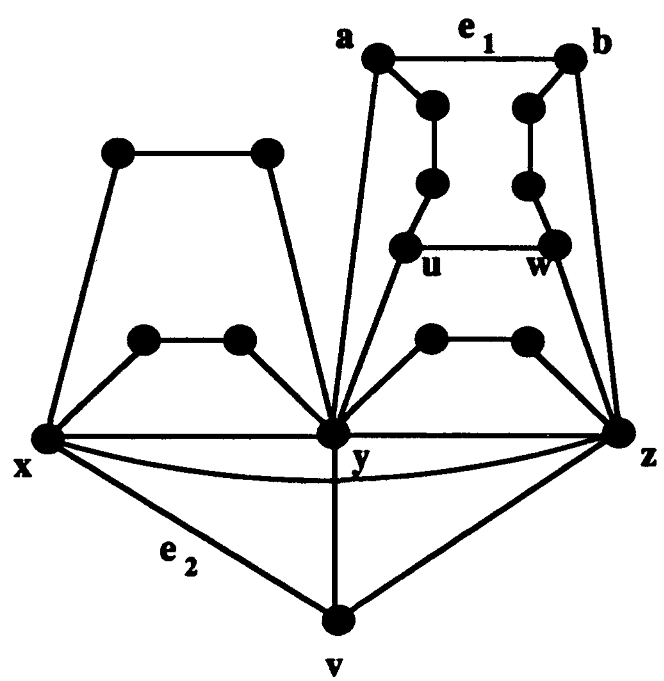


Figure 5.14 An edge becomes λ_i-neutral after the removal of another λ_i-neutral edge.

Now that it has been established that a non-neutral edge can become neutral with the removal of another neutral edge, it is a natural consequence to turn our attention to what changes may occur in the total number of neutral edges in a graph during edge deletions. To verify the examples, the computer software from [12] was again used.

We begin this investigation with the previous figure. Here, there are a total of nine λ_i-neutral edges: ab, uw, vy, vz, yz, yu, ya, zw, and zb. After the removal of e_1, the only other edge to become λ_i-neutral is e_2. But there are four edges that lose their λ_i-neutrality: ya, zb, uw, and yz. The deletion of edge uw or yz from G - e_1 causes the λ_i value for v to decrease by one since there would now only be three edge disjoint yz paths. The deletion of edge ya or zb from G - e_1 leaves a or b as an endvertex making its only neighbor a cutvertex. Thus, there are only five λ_i-neutral edges in G - e_1, a decrease of four such edges from G.

However, we cannot assume that the total number of λ_i-neutral edges in a graph will not strictly increase with the deletion of a λ_i-neutral edge. This surprising result can be observed in Figure 5.15. This figure also demonstrates the case where the λ_i value initially decreases by one after the deletion of the edge e_2 which is not λ_i-neutral, which implies that the location of e_1 and e_2 in relation to v must be reversed from Figure 5.14.

In the graph of Figure 5.15 the λ_i values for every vertex except v are equal to one in G, G - e_1, G - e_2, and G - e_1 - e_2. But $\lambda_i(v, G) = \lambda_i(v, G - e_1) = \lambda_i(v, G - e_1 - e_2) = 5$ and $\lambda_i(v, G - e_2) = 4$. Note that there are five edge disjoint paths between every pair of neighbors of v in G - v. In G - e_1 the λ_i value for v must be computed from the number of edge disjoint x-y paths, upon which e_2 has no bearing. Thus, e_2 is λ_i-neutral in G - e_1. But there are only four edge disjoint yz paths in G - e_2 which implies e_2 is not λ_i-neutral in G.

Using the fact there are five edge disjoint paths between every pair of neighbors of v in G - v, we can locate the three λ_i-neutral edges in G: vx, vy, and vz. In G - e_1 only the edge vx loses its λ_i-neutrality, since λ_i(y, G - e_1 - vx) = 8. But seven edges become λ_i-neutral: yz, yu, ya, zw, zb, uw, and ab. The endpoints of each of these edges acquire their λ_i value of one from an adjacent vertex of degree two. Also, the removal of one of these edges will not change the number of remaining edge disjoint xy paths. Thus the removal of any one of these edges will not change the λ_i value of any vertex. Therefore, the number of λ_i-neutral edges increases from 3 in G to 8 in G - e_1!

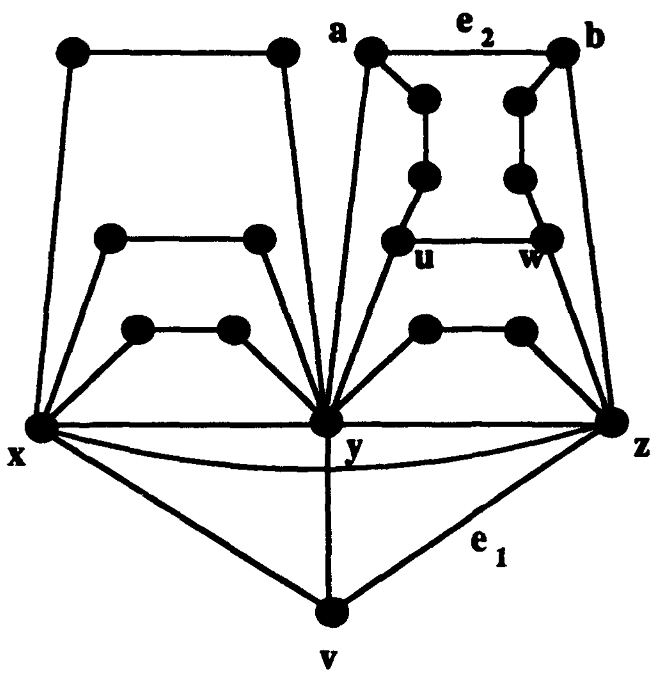


Figure 5.15 The total number of λ_i-neutral edges increase after the deletion of e_1.

After observing the change in the number of λ_i-neutral edges in the previous figure, we see from the graph in Figure 5.16 that the increase in the number of λ_i-neutral edges can be arbitrarily large. Each similar duplication of the section

containing vertices a, b, u, and w with an additional pair of edge disjoint xy paths will increase the number of λ_i-neutral edges in G - e_1 by six.

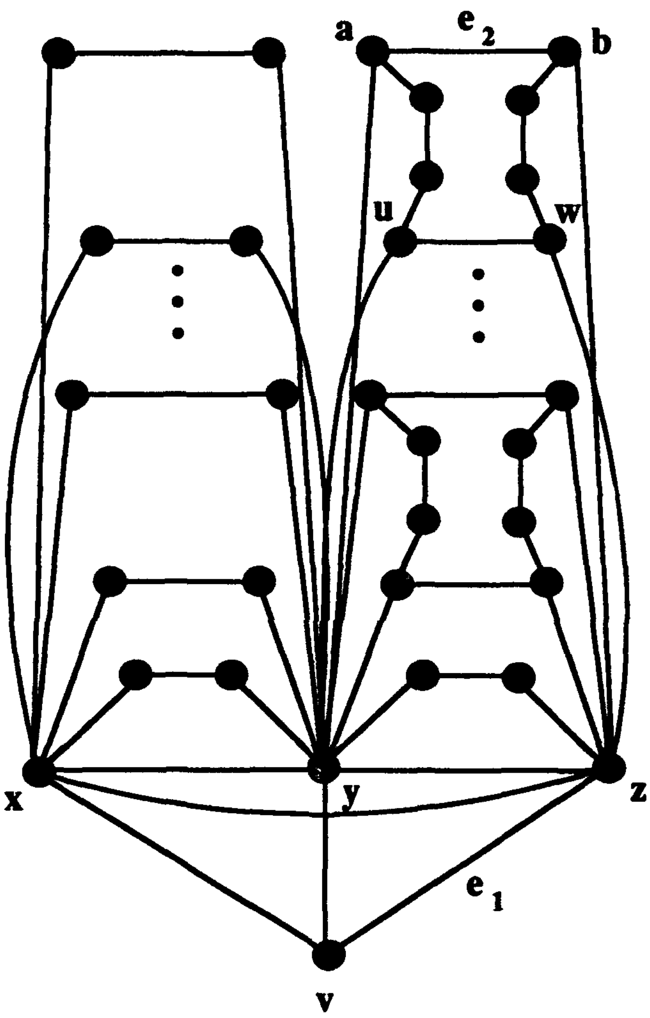


Figure 5.16 The increase in the number of λ_i-neutral edges can be arbitrarily large.

Stable Graphs for Inclusive Connectivity

We define the λ_i value of a graph G, $\lambda_i(G)$, to be the sum of all the λ_i values of the vertices of G. Likewise, we define $\kappa_i(G)$ ($\mu_i(G)$) to be the sum of all the κ_i (μ_i) values of the vertices of G.

Clearly, a λ_i, κ_i, or μ_i-neutral edge does not change $\lambda_i(G)$, $\kappa_i(G)$, or $\mu_i(G)$ respectively. Any edge e that has the property that $\lambda_i(G) = \lambda_i(G - e)$ ($\kappa_i(G) = \kappa_i(G - e)$, $\mu_i(G) = \mu_i(G - e)$) is called *λ_i- (κ_i, μ_i) s-stable* which is an abbreviation for sum-stable. And finally, a graph G is called *λ_i- (κ_i, μ_i) stable* if $\lambda_i(G) = \lambda_i(G - e)$ ($\kappa_i(G) = \kappa_i(G - e)$, $\mu_i(G) = \mu_i(G - e)$) for every $e \in E(G)$. Simply put, a graph is λ_i, κ_i, or μ_i-stable if the sum of the respective λ_i, κ_i, or μ_i values remains the same no matter what edge is deleted. This definition of stability for a graph is natural when one notices that the average i-connectivity of the vertices of the graph remain the same when any edge is deleted.

We begin this section with an investigation into a special case of neutral edges for inclusive connectivity which will provide us with the first example of a type of stable graph.

Since every nontrivial graph (except K_2) contains at least two vertices that are not cutvertices [7], at least two vertices have nonempty λ_i-sets. This implies that in any nontrivial graph except K_2 there exists an edge that is not λ_i-neutral since the deletion of any edge in a λ_i-set for a vertex will decrease the λ_i value for that vertex by one. Similarly, if any μ_i-set for any vertex in a graph contains an edge, then there exists an edge that is not μ_i-neutral.

We now investigate the possibility that *every* edge in a graph is κ_i-neutral. Also, if every μ_i-set for every vertex in a graph is also a κ_i-set, it is possible for every edge in the graph to be μ_i-neutral. For the existence of this possibility, we refer to the graph in Figure 5.17 and Table 5.1. Due to symmetry, every possible change in the i-connectivity values is represented by the edge deletions in this table.

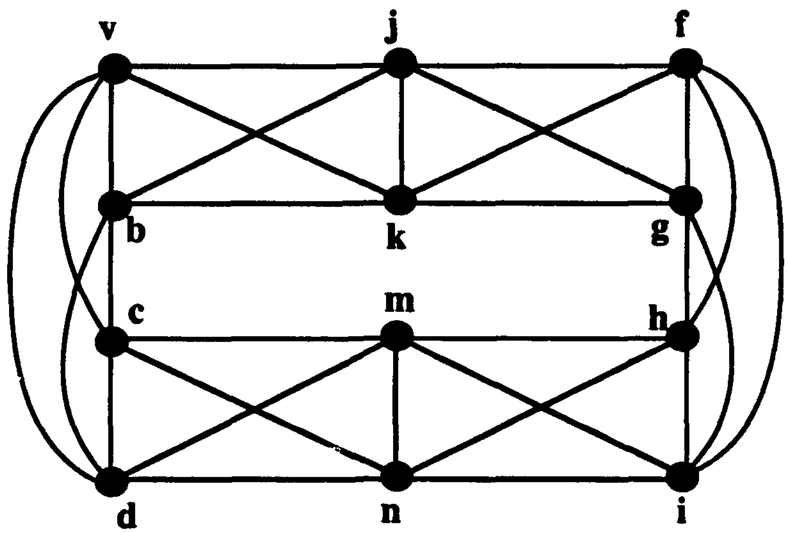


Figure 5.17 Every edge is κ_i-neutral and μ_i-neutral.

Table 5.1 The changes in i-connectivity values from the graph in Figure 5.17.

	G			G - vj			G - vk			G - jk			G - vb			G - vc			G - vd		
Vertex	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i
v	3	4	3	3	4	3	3	4	3	3	3	3	3	4	3	3	4	3	3	4	3
b	3	4	3	3	3	3	3	3	3	3	3	3	3	4	3	3	3	3	3	3	3
c	3	4	3	3	3	3	3	3	3	3	4	3	3	3	3	3	4	3	3	3	3
d	3	4	3	3	3	3	3	3	3	3	4	3	3	3	3	3	3	3	3	4	3
f	3	4	3	3	3	3	3	3	3	3	3	3	3	4	3	3	4	3	3	4	3
g	3	4	3	3	3	3	3	3	3	3	3	3	3	4	3	3	4	3	3	4	3
h	3	4	3	3	4	3	3	4	3	3	4	3	3	4	3	3	4	3	3	4	3
i	3	4	3	3	4	3	3	4	3	3	4	3	3	4	3	3	4	3	3	4	3
j	3	4	3	3	4	3	3	3	3	3	4	3	3	3	3	3	3	3	3	3	3
k	3	4	3	3	3	3	3	4	3	3	4	3	3	3	3	3	3	3	3	3	3
m	3	4	3	3	4	3	3	4	3	3	4	3	3	4	3	3	3	3	3	3	3
n	3	4	3	3	4	3	3	4	3	3	4	3	3	4	3	3	3	3	3	3	3

For any vertex $u \in V(G)$, there are no fewer than three internally disjoint paths between a pair of its nonadjacent neighbors in $G - u$ and $G - u - e$, for every $e \in E(G)$. So $\kappa_i(u, G) = \mu_i(u, G) = \kappa_i(u, G - e) = \mu_i(u, G - e) = 3$ for every vertex $u \in V(G)$ and edge $e \in E(G)$. For example, a κ_i-set S_v for v in G is { b, m, n } or { b, f, g } where the vertices d, $k \in N_G(v)$ are separated in $G - S_v - v$. Therefore, every edge in this figure is κ_i-neutral as well as μ_i-neutral!

Since the existence of this type of κ_i-neutrality is established, the following theorem is immediate.

Theorem 5.6: Every edge of a graph G is κ_i-neutral if and only if every vertex in G is κ_i-stable under edge deletion.

Proof: Every edge of G is κ_i-neutral if and only if $\kappa_i(v, G) = \kappa_i(v, G - e)$ for all $e \in E(G)$ and for any vertex $v \in V(G)$ if and only if every vertex in G is κ_i-stable under edge deletion. ❑

A similar result for μ_i is presented.

Theorem 5.7: Every edge of a graph G is μ_i-neutral if and only if every vertex in G is μ_i-stable under edge deletion.

It is interesting to point out that the graph in Figure 5.17 is the first known example of a graph that does not have a neighborhood κ_i or μ_i-set for any vertex. This idea is formalized in the next theorem.

Theorem 5.8: If every edge in a graph G is κ_i- (μ_i) neutral, then G has no neighborhood κ_i- (μ_i) sets for any vertex.

Proof: If G has a neighborhood κ_i- (μ_i) set for some vertex $v \in V(G)$, then select for deletion any edge whose endpoints are that neighbor of v and some vertex in the κ_i- (μ_i) set. (If the μ_i-set does not contain any vertices, any edge in the μ_i-set will suffice.) This will cause the respective i-connectivity value for v to decrease by one, implying that selected edge is not κ_i- (μ_i) neutral. ❑

Note that the regular graph in Figure 5.17 has $\delta(G) = 5$ which is strictly greater than the κ_i or μ_i value for any vertex, which must be true as a consequence of Theorem 5.8. Another result of this theorem is that a graph whose every edge is κ_i or μ_i-neutral cannot contain any vertices of degree one, since they rely exclusively on a neighborhood set. This idea can be extended to vertices of degree two as Corollary 5.10 below shows.

Theorem 5.9: If $v \in V(G)$ has degree one or two, then v is not κ_i or μ_i-stable under edge deletion.

Proof: Let $v \in V(G)$ have degree one. Since every κ_i and μ_i-set for v is a neighborhood κ_i, μ_i-set, then v is not κ_i or μ_i-stable under edge deletion.

Let $deg_G(v) = 2$ where $u, w \in N_G(v)$, $u \neq w$. If there exists a neighborhood κ_i-set for v in G, then v is not κ_i-stable under edge deletion. So assume there does not exist any neighborhood κ_i-sets for v in G. Then $\kappa_i(v, G) < \min\{ deg_{G-v}(u), deg_{G-v}(w)\}$. So $\kappa_i(v, G - vu) = deg_{G-v}(w) > \kappa_i(v, G)$ implying v is not κ_i-stable under edge deletion in G. The argument for μ_i is identical. ❑

The proof of the corollary is immediate and omitted.

Corollary 5.10: If every edge of G is κ_i or μ_i-neutral, then $\delta(G) \geq 3$.

The special case of neutrality displayed in the graph of Figure 5.17 provides us with the first example of the different types of stable graphs. Theorems 5.6 and 5.7 imply that this graph is κ_i-stable and μ_i-stable since no κ_i and μ_i values for any vertex change with the deletion of any edge.

In this graph, $\lambda_i(d, G) = 4$ for every vertex $d \in V(G)$. However, the λ_i values for several vertices in this figure decrease with the deletion of a particular edge. For example $\lambda_i(x, G - vb) = 3$ with a λ_i-set for x in G - vb consisting of the neighborhood set from the vertex b. No λ_i values increase with the deletion of the edge vb, so the sum of all λ_i values decreases, implying G is not λ_i-stable. In fact, $\lambda_i(G) = 48 > 42 =$

$\lambda_i(G - vb)$. Thus, the graph of Figure 5.17 is the first example of a κ_i-stable and μ_i-stable but not λ_i-stable graph.

This example can be expanded to an infinite class of graphs as in Figure 5.18.

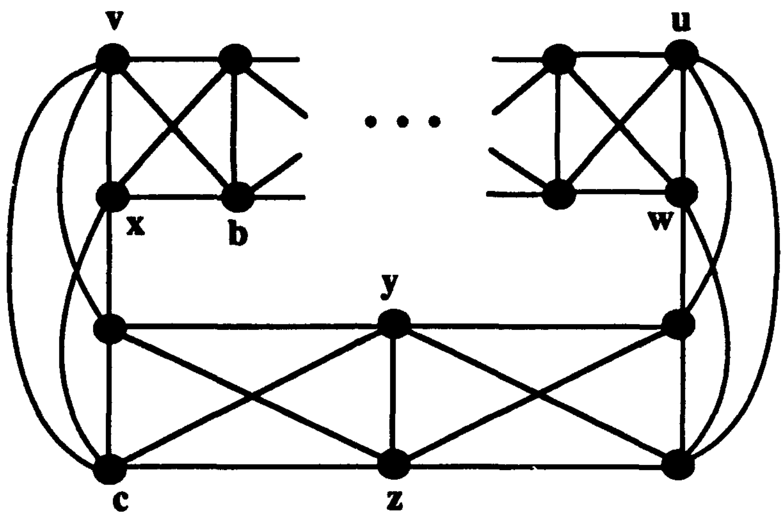


Figure 5.18 An infinite class of κ_i- and μ_i-stable but not λ_i-stable graphs.

The vertices added to the graph of Figure 5.17 to form Figure 5.18 do not create any larger sets of internally disjoint or edge disjoint paths between neighbors of any vertex, and they will have i-connectivity values the same as vertex b. Any previous paths that transit the intermediate vertices (like vertex b) on the upper tier will be preserved but with a greater length. Thus the graph in Figure 5.18 shares i-connectivity stability with the one in Figure 5.17.

Ringeisen and Rice [21] were the first to investigate λ_i-stable graphs. One of their first λ_i-stable graphs is shown in Figure 5.19. The label K_6 - E means a K_6 from which a one-factor, E, has been removed. All λ_i values in G are equal to two so $\lambda_i(G) = 36$. Removal of the edge uv results in the λ_i values of only three vertices changing

as shown in the graph of Figure 5.20; removal of the edge vw results in the λ_i values of six vertices changing as shown in the graph of Figure 5.21. These are the only types of edge deletions which cause a change in λ_i values, and $\lambda_i(G) = \lambda_i(G - uv) = \lambda_i(G - vw) = 36$ implying λ_i-stability. Upon further investigation of the graphs in Figures 5.19 - 5.21, the same changes occur for both κ_i and μ_i values, meaning Figure 5.19 gives an example of a λ_i, κ_i, and μ_i-stable graph.

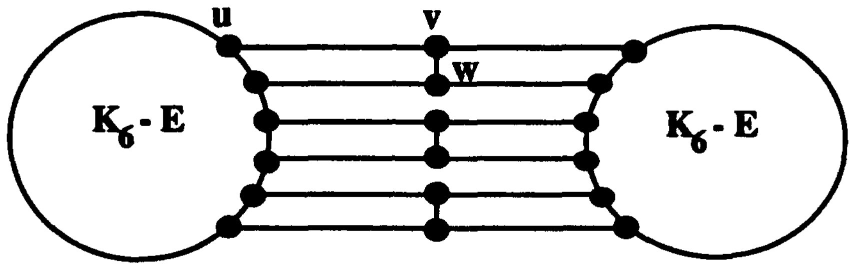


Figure 5.19 A λ_i, κ_i and μ_i-stable graph.

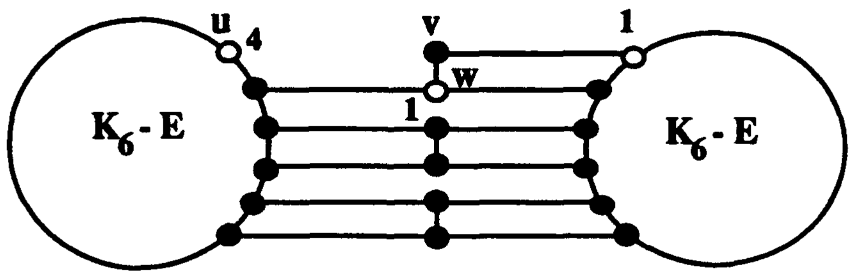


Figure 5.20 The change in λ_i values in G - uv.

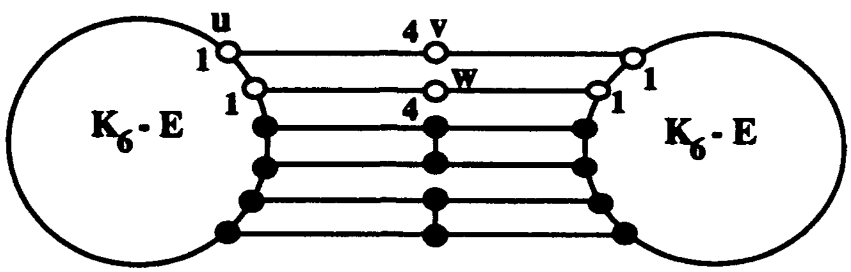


Figure 5.21 The change in λ_i values in G - vw.

To obtain an infinite class of λ_i, κ_i, and μ_i-stable graphs, the reader is referred to Rice's work [18] establishing the existence and construction of λ_i-stable graphs.

Another example of a λ_i-stable graph found by Rice that is also κ_i and μ_i-stable is illustrated in the graph of Figure 5.22. The subgraph of this figure consisting of a "K_4 with one edge doubly subdivided" is an extremely useful form in obtaining various types of stable graphs which will be present in the examples concluding this section.

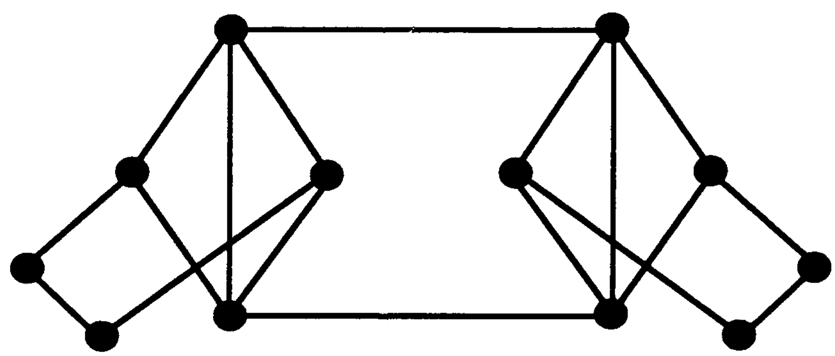


Figure 5.22 Another λ_i, κ_i, and μ_i-stable graph.

For a complete graph K_n, $n \geq 3$, all i-connectivity values are equal to their maximum value of n - 2 which can be attained by the use of neighborhood i-connectivity sets. But with the deletion of any edge of K_n, any vertex adjacent to the

endpoints of that edge will have all i-connectivity values decrease by one. Thus, K_n is not λ_i, κ_i, or μ_i-stable. Another graph of this type is given in Figure 5.23. Here, $\lambda_i(G) = 4$ while $\lambda_i(G - e) = 2$, for any edge e.

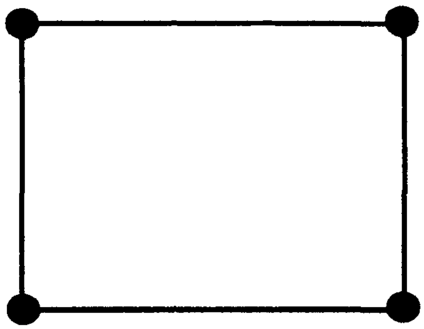


Figure 5.23 A graph that is not λ_i, κ_i, or μ_i-stable.

Using a construction technique based on Rice's use of the "K_4 with one edge doubly subdivided", we are able to obtain three types of stable graphs. This technique involves the attachment of copies of this subgraph to an internal graph G in a manner that will achieve the desired stability. For simplicity in this section, we will call this graph G the "internal G graph" due to its location in the figures, and the "K_4 with one edge doubly subdivided" subgraph the "subdivided K_4". For example, in Figure 5.24, the internal graph G is the graph K_4 which has pairs of its vertices attached to one copy of the subdivided K_4 producing the graph in Figure 5.25.

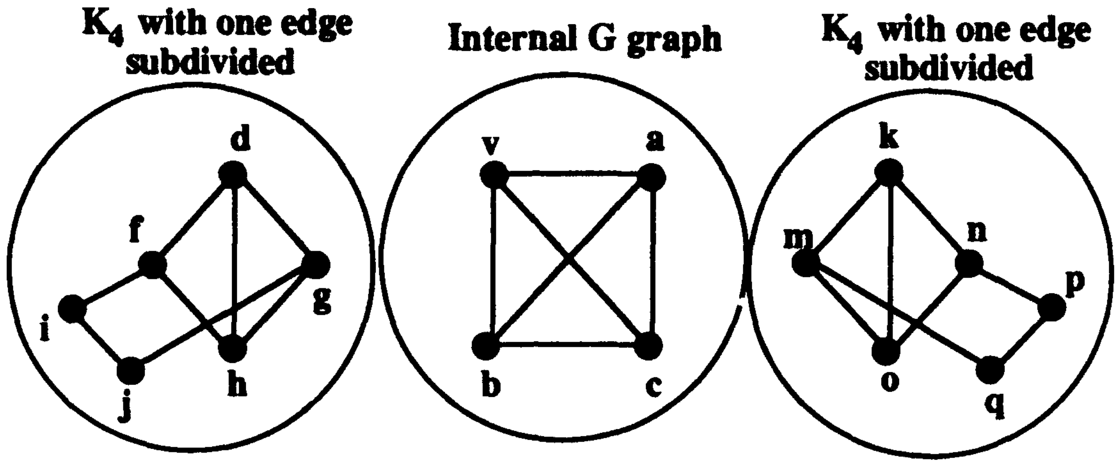


Figure 5.24 The construction technique of the "K_4 with one edge subdivided".

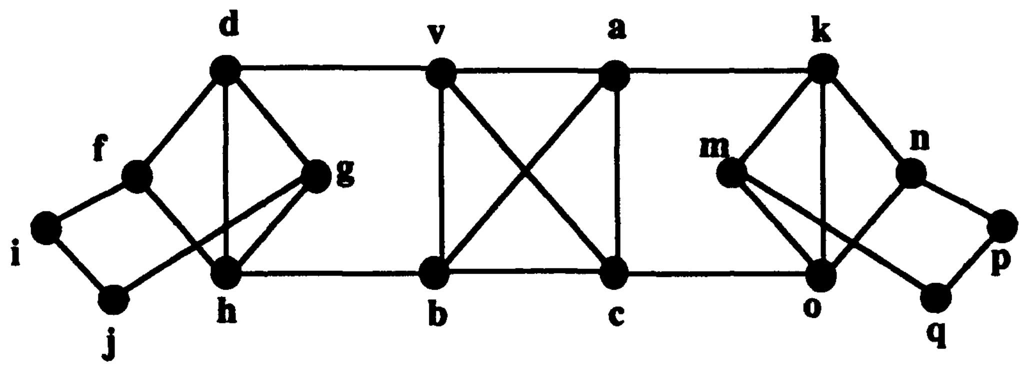


Figure 5.25 A graph that is λ_i and μ_i-stable but not κ_i-stable.

A summary of the changes in the i-connectivity value after edge deletion is presented in Table 5.2. The i-connectivity values for every vertex in this graph is equal to one. These values remain one with the removal of any edge from the K_4 subgraph. The removal of any edge from either subdivided K_4 changes the i-connectivity values in a manner similar to the graph in Figure 5.22 so that the overall sum of the λ_i values remains unchanged. The critical edges that produce the desired stability properties are the connecting edges between these subgraphs. For instance,

the removal of edge vd forces the vertex v to obtain its i-connectivity values from the basic structure of the internal G graph.

Note that the neighborhood of v in G - vd is complete, so $\kappa_i(v, G - vd) = 3$. But $\lambda_i(v, G - vd) = \mu_i(v, G - vd) = 2$ with a λ_i-set being { bc, ab } and a μ_i-set is { bc, a }. Vertices b and h become cutvertices in G - vd while the i-connectivity values for vertex d increase to two. Thus, vertices b, h, and d account for a overall decrease of one for the i-connectivity sum of the graph. In order to preserve stability, there must be an increase of exactly one from the vertex v which is the only other vertex to have a change in its i-connectivity values. Therefore, this graph is λ_i and μ_i-stable but not κ_i-stable.

Table 5.2 The changes in i-connectivity values from the graph in Figure 5.25.

	G			G - va, G - vb			G - vc, G - dg			G - gh, G - dh			G - vd			G - fi			G - ij		
Vertex	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i
v	1	1	1	1	1	1	1	1	1	1	1	1	3	2	2	1	1	1	1	1	1
a	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
b	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1
c	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
d	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	1	1	1	1	1	1
f	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	3	3	3	0	0	0
g	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0
h	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1
i	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2
j	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	2	2	2
k	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
m	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
n	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
o	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
p	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
q	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

This construction technique will also produce a κ_i and μ_i-stable but not λ_i-stable graph similar to that in Figure 5.17. The i-connectivity values for every vertex in the graph of Figure 5.26 are equal to one. But removal of edge vg or fq causes $\lambda_i(G)$ to change.

Let us denote the graph in Figure 5.26 as G. The changes in the i-connectivity values for G are displayed in Table 5.3. S in this table includes the following: any one of {∅, va, vc, vf, ab, ac, ad, bd, bf, cd, and cf}. In (G - vg) - v there are a maximum of three edge disjoint ca, cf, and fa paths so $\lambda_i(v, G - vg) = 3$. But vertices a and h become cutvertices while the i-connectivities for vertex g increase to two. So $\lambda_i(G) = 24 \neq 25 = \lambda_i(G - vg)$. Thus, this graph is not λ_i-stable while the sums of the κ_i and μ_i values remain unchanged at 24 with the deletion of any edge.

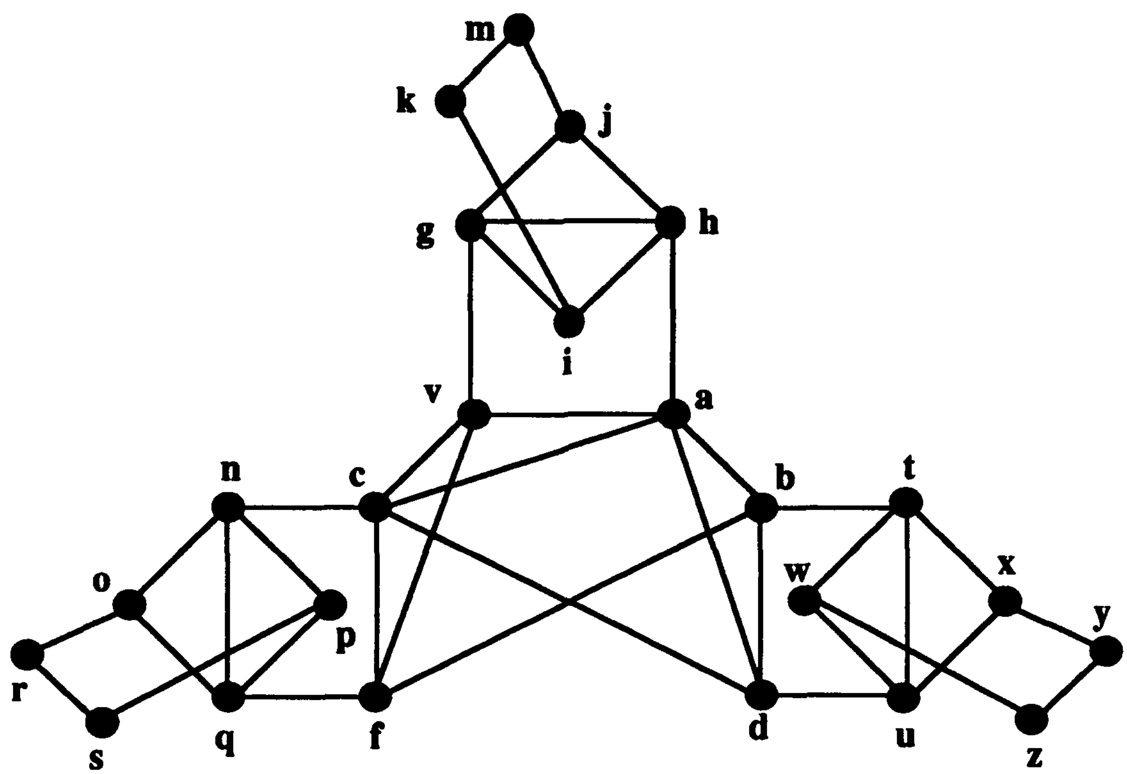


Figure 5.26 A graph that is κ_i- and μ_i-stable but not λ_i-stable.

Table 5.3 The changes in i-connectivity values from the graph in Figure 5.26.

	G - S			G - vg			G - ah			G - bt			G - cn			G - du			G - fq		
Vertex	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i
v	1	1	1	2	3	2	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
a	1	1	1	0	0	0	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1
b	1	1	1	1	1	1	1	1	1	2	2	2	1	1	1	0	0	0	1	1	1
c	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	1	1	1	0	0	0
d	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	2	2	2	1	1	1
f	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	2	3	2
g	1	1	1	2	2	2	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
h	1	1	1	0	0	0	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1
i	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
j	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
k	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
m	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
n	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	1	1	1	0	0	0
o	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
p	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
q	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	2	2	2
r	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
s	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
t	1	1	1	1	1	1	1	1	1	2	2	2	1	1	1	0	0	0	1	1	1
u	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	2	2	2	1	1	1
w	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
x	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
y	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
z	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

With this construction technique, it is clear that initially all i-connectivity values are equal to one. After the removal of a edge between the internal G graph and a subdivided K_4 subgraph, the vertex incident with this edge in the internal G graph is now forced into the internal G graph for its i-connectivity value and must have a value equal to two to preserve stability. This is due to two vertices becoming cutvertices and the value for one vertex increasing to two. Thus, the structure of the internal G graph holds the key to what stabilities are achieved. Also, the pairing of the vertices of the internal G graph to the other subgraphs can play a key role in not having an

increase in the maximum number of edge disjoint or internally disjoint paths due to an additional path through the subdivided K_4 subgraph.

The last example presented in this chapter is a graph that is λ_i-stable but not κ_i or μ_i-stable. The internal G graph used in this example is shown in Figure 5.27. The i-connectivity values for every vertex in this graph are equal to two except for the κ_i and μ_i values for vertices a and b which are equal to one. A κ_i and μ_i-set for b is { a } and vice versa.

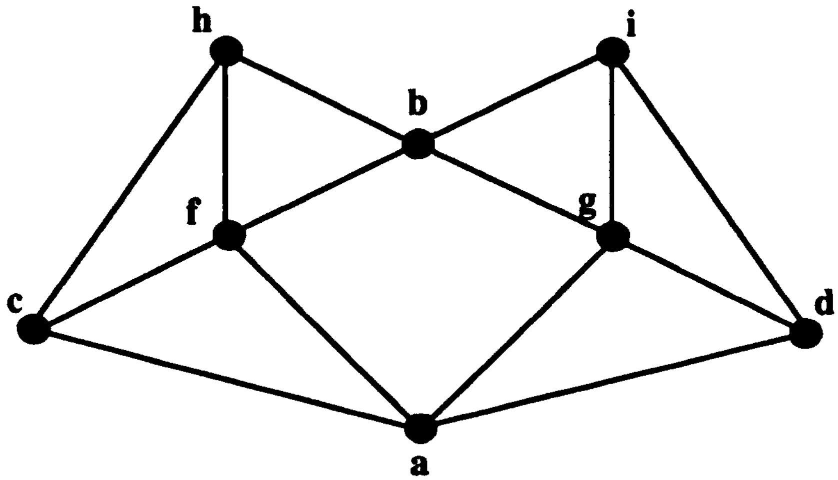


Figure 5.27 An internal G graph that produces the desired stability.

The graph H, constructed from G, with the desired i-connectivity stability is illustrated in Figure 5.28. Table 5.4 includes the changes in i-connectivity from edge deletion. S in this table includes the following: any one of { ∅, ad, af, ag, bf, bg, bh, cf, ch, di, and gi.

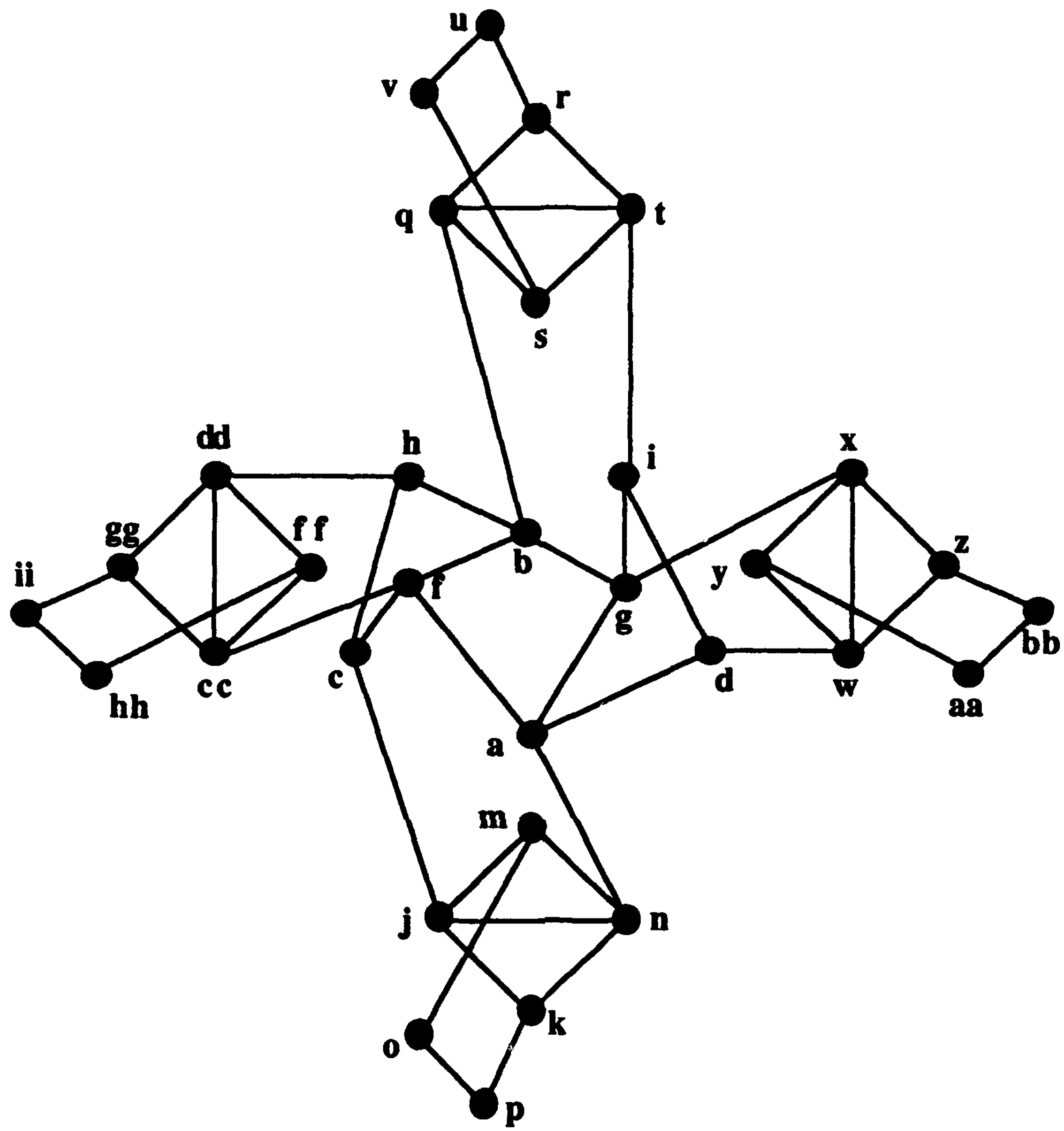


Figure 5.28 A graph that is λ_i-stable but not κ_i or μ_i-stable.

Note that the edges ac, bi, dg, and fh in the graph of Figure 5.27 were eliminated in Figure 5.28 since these edges were essentially replaced by the subdivided K_4 subgraphs. The deletion of edge ax or bcc results in $\kappa_i(a, G - ax) = \mu_i(a, G - ax) = 1$ or $\kappa_i(b, G - (bcc)) = \mu_i(b, G - (bcc)) = 1$. A κ_i and μ_i-set for a in G - ax is { b } while a κ_i and μ_i-set for b in G - (bcc) is { a }. Thus, with the known i-connectivity changes

that occur as a result of the construction technique, G is not κ_i or μ_i stable. Due to the structure of the internal graph and the pairing of its vertices with elimination of unneeded edges, λ_i-stability is achieved.

Table 5.4 The changes in i-connectivity values from the graph in Figure 5.28.

	G - S			G - an			G - bq			G - cj			G - dw			G - fcc			G - gx			G - hdd			G - it		
Vertex	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i	κ_i	λ_i	μ_i
a	1	1	1	1	2	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
b	1	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0
c	1	1	1	0	0	0	1	1	1	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
d	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	1	1	1	0	0	0	1	1	1	1	1	1
f	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	1	1	1	0	0	0	1	1	1
g	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	2	2	2	1	1	1	1	1	1
h	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	2	2	2	1	1	1
i	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2
j	1	1	1	0	0	0	1	1	1	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
k	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
m	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
n	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
o	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
p	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
q	1	1	1	1	1	1	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0
r	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
s	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
t	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2
u	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
v	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
w	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	1	1	1	0	0	0	1	1	1	1	1	1
x	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	2	2	2	1	1	1	1	1	1
y	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
z	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
aa	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
bb	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
cc	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	1	1	1	0	0	0	1	1	1
dd	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	2	2	2	1	1	1
ff	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
gg	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
hh	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ii	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

CHAPTER 6

CONCLUSION

Introduction

This chapter begins with the presentation of some extensions of a powerful theorem from [2]. These extensions could prove beneficial to resolving the one remaining case concerning interrelationships from Chapter 3.

We conclude with some conjectures on open problems and ideas for future research involving inclusive connectivity.

Extensions of Boland's One Edge Theorem

The first section of this chapter deals with extensions of Theorem 3.19, Boland's "One Edge Theorem". These results stem from our unsuccessful efforts to prove that a vertex that is λ_i and μ_i-stable, must not be κ_i-stable under edge addition. It appears that Theorem 3.19 will play a critical role in this proof and a better understanding of the conditions surrounding it is thus warranted.

First, we deal with the location of the neighbors of a vertex $v \in V(G)$ that are not elements of the separated pair.

Theorem 6.1: If S_m is a μ_i-set (as described in Theorem 3.19) that contains exactly one edge and that edge has as its endpoints the neighbors u and w of $v \in V(G)$, then any other neighbors of v in G must be contained in S_m.

Proof: Let $v \in V(G)$ where $\mu_i(v) < \kappa_i(v)$. Let S_m be a μ_i-set for v detailed in Theorem 3.19 which contains exactly one edge where that edge is between u, $w \in N(v)$. Assume for the sake of obtaining a contradiction that there exists a vertex $x \in N(v)$ where $x \neq u$, w and $x \notin S_m$. Then S_m contains exactly one edge, namely uw, whose endpoints are neighbors of v, and by Theorem 3.20, there are exactly two

components of G - S_m - v which contain vertices of N(v), say C_1 and C_2. Without loss of generality let u, x $\in$ V(C_1) and w $\in$ V(C_2).

Since S_m contains only one edge uw, and since x and w are in components C_1 and C_2 respectively, then xw $\notin$ E(G). Thus S_m - uw + {u } would be a set of vertices of cardinality $\mu_i(v)$ that separates neighbors x and w of v. Combining this with the result $\mu_i(v) \leq \kappa_i(v)$, makes S_m - uw + { u } a κ_i-set, contradicting $\mu_i(v) < \kappa_i(v)$. ❑

An interesting corollary now follows.

Corollary 6.2: For any v $\in$ V(G), if $\mu_i(v) < \kappa_i(v)$, then there exists a set of $\mu_i(v)$ internally disjoint paths between two neighbors of v, so that every other neighbor of v is on one of the paths and no such path contains more than one neighbor of v.

Proof: Let u and w be neighbors of v separated by $\mu_i(v)$ internally disjoint paths. From Theorem 6.1, we know every neighbor of v besides u and w is in the μ_i-set U. By Menger's Theorem, we know there must be $\mu_i(v)$ internally disjoint u-w paths and thus any x $\in$ N(v) where x $\neq$ u, w must be on one of the paths. Moreover if one path contains two neighbors besides u and w then one of them does not have to be in U, a contradiction. ❑

Open Problems

The previously mentioned unresolved case of a vertex that is λ_i and μ_i-stable but not κ_i-stable under edge addition has led to Conjecture 6.3, implying this case fails to exist.

Conjecture 6.3: If v $\in$ V(G) is λ_i and μ_i-stable under edge addition, then v is κ_i-stable under edge addition.

This case leads us to propose two other related conjectures.

Conjecture 6.4: If v $\in$ V(G) is μ_i-stable under edge addition, then $\mu_i(v) = \kappa_i(v)$.

Conjecture 6.5: If v $\in$ V(G) is λ_i-stable under edge addition, then $\lambda_i(v) \geq \kappa_i(v)$.

Conjecture 6.5 would provide another situation similar to Theorem 3.14 implying that a vertex would "behave normally" under λ_i-stability.

The discovery of the relationship between the stability of inclusive connectivity and the stability of the global connectivities under edge addition was presented in Chapter 3. The establishment of this interesting relationship could provide an alternate door for the study of the global connectivity parameters and deserves much further study.

Continued study into inclusive connectivity stable graphs could provide examples of all possible combinations of stability. Expansion of the examples to infinite classes similar to Rice's work could provide insight into the possible structure of these graphs.

Finally, the relationship between i-connectivity stable graphs and neutral edges has not been addressed. In particular, is there a relationship between the number of neutral edges in a specific structure and stability?

Clearly, the opportunities of further research into inclusive connectivity seem to be promising and bright.

REFERENCES

1. L.W. Beineke and F. Harary, The Connectivity Function of a Graph, *Mathematika*, 1967, **14**, 197-202.

2 J.W. Boland, Inclusive Connectivity, A Local Graph Connectivity Parameter, Doctoral Dissertation (Clemson University, Clemson, SC, 1991).

3. J.W. Boland and R.D. Ringeisen, Inclusive Connectivity: An Introduction to a Loc Parameter, *Congr. Numer.*, 1990, **78**, 217-221.

4. J.W. Boland and R.D. Ringeisen, Relationships Between i-Connectivity Parameters, Advances in Graph Theory, Vishwa International Publications, Gulbarga, India, 1991, 33-40.

5. G. Chartrand, A Graph-Theoretic Approach to a Communications Problem, *J. SIAM Appl. Math.*, 1966, **14**, 5, 778-781.

6. G. Chartrand and F. Harary, Graphs with Prescribed Connectivities in Theory of Grapfs, P. Erdős, G. Katona, Eds. Akademiai Kiado, Budapest, 61-63, 1968.

7. G. Chartrand and L. Lesniak, Graphs & Digraphs, Wadsworth & Brooks/Cole, Monterey CA., 1986.

8. G. Chartrand and R.E. Pippert, Locally connected graphs, *Casopis Pest. Math.*, 1974, **99**, 158-163.

9. D.W. Cribb, J.W. Boland and R.D. Ringeisen, Graphs with Stable Inclusive Connectivity Parameters, submitted.

10. Lai Hong-Jian and Lai Hongyuan, preprint, The Size of Graphs with Given Inclusive Connectivity, 1991.

11. Z. Huai-lu, A Necessary and Sufficient Condition for Connected, Locally k-Connected $K_{1,3}$-Free Graphs to be k-Hamiltonian, *J. Graph Theory*, 1989, **13**, 6, 651-656.

12. C. Lee, Computing Inclusive Connectivity, Master's Paper (Clemson University, Clemson, SC, 1990).

13. L. Lesniak, Results on the Edge-connectivity of Graphs, *Discrete Math.*, 1974, **8**, 351-354.

14. M.J. Lipman and R.D. Ringeisen, Cohesion in Alliance Graphs, *Congr. Numer.*, 1979, **24**, 713-720.

15. W. Mader, Degree and local connectivity in digraphs, Combinatorica, 1985, **5**, 2, 161-165.

16. D.J. Oberly and D.P. Sumner, Every Connected, Locally Connected Nontrivial Graph with No Induced Claw is Hamiltonian, *J. Graph Theory*, 1979, **3**, 351-356.

17. K.B. Reid, Remarks on Cohesion and Induced Edge Connectivity, *Congr. Numer.*, 1983, **38**, 253-259.

18. V. Rice, Cohesion Properties in Graphs, Doctoral Dissertation (Clemson University, Clemson, SC, 1988).

19. V. Rice and R.D. Ringeisen, Cohesion stable edges, *Congr. Numer.*, 1988, **67**, 205-209.

20. R. Ringeisen and V. Rice, Cohesion stability under edge destruction, *J. Comb. Math. and Comb. Computing.*, 1988, **2**, 153-162.

21. R. Ringeisen and V. Rice, When is a graph not stable or are there any stable graphs out there ?, *J. Comb. Math. and Comb. Computing.*, 1988, **4**, 19-22.

22. R.D. Ringeisen and M.J. Lipman, Cohesion and Stability in Graphs, *Discrete Math.*, 1983, **46**, 253-259.

23. D. Vanderjagt, Graphs with prescribed local connectivities, *Discrete Math.*, 1974, **10**, 391-395.

24. D. Vanderjagt, Sufficient conditions for locally connected graphs, *Casopis Pest. Math.*, 1974, **99**, 400-404.

25. H. Whitney, Congruent Graphs and the Connectivity of Graphs, *Amer. J. Math.*, 1932, **54**, 150-168.

26. S.S. Yau, A generalization of the cutset, *J. Franklin Inst.*, 1962, **273**, 31-48.